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Mathematical Reviews

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FOUNDATIONS

*Curry, Haskell B. *L-semantics as a formal system.* Congrès International de Philosophie des Sciences, Paris, 1949, vol. II, Logique, pp. 19-29. *Actualités Sci. Ind.*, no. 1134. Hermann & Cie., Paris, 1951.

A formal system is constructed in six successive stages which can be summed up as follows. (1) A formal system is set up which describes a Boolean algebra R (the elements of which correspond to L -ranges, that is, classes of L -states [cf. R. Carnap, *Introduction to semantics*, Harvard Univ. Press, 1942, and *Formalization of logic*, *ibid.*, 1943; these *Rev.* 4, 209]. (2) This formal system is enlarged so as to allow description of a Boolean algebra T (the elements of which correspond to classes of sentences). (3) The next step is to provide for the description of the assignment of an element of R to each element of T . (4) Then provisions are made for dealing with individual sentences. (5)-(6) Finally, the system is equipped with the means to describe properties of truth table connectives and to describe the special peculiarities of Carnap's language S_1 . The construction of the formal system is carried out, of course, without reference to its purpose of providing a formalization of Carnap's L -semantics. But it is pointed out that the formal system resulting from (1)-(6) admits the interpretation of being a description of S_1 , and that already at stage (3) all postulates in Carnap's L -semantics, except 8, 11, 12, 13, can be derived, these latter being derivable at stage (4). On the other hand, even (1)-(6) does not exhaust the aforementioned interpretation, it admits various extensions, some of which are consistent with this interpretation while others disagree with it. It is observed furthermore that the formal system resulting from (1)-(6) can be incorporated in the first order predicate calculus.

A more detailed metamathematical investigation of Curry's formal system will probably prove interesting. It will be clear that the system (1)-(3) as well as any of its extensions determines a relationship between the isomorphism types of the Boolean algebras R and T . It would be interesting to know exactly how this relationship depends on the manner in which this system is extended. It would also be desirable to extend the system so as to provide for the treatment of quantifiers.

E. W. Beth.

Curry, Haskell B. *A new proof of the Church-Rosser theorem.* *Nederl. Akad. Wetensch. Proc. Ser. A* 55 = *Indagationes Math.* 14, 16-23 (1952).

A theorem is proved in the general theory of equivalence relations defined by "moves", developed by the reviewer [*Ann. of Math.* (2) 43, 223-243 (1942); these *Rev.* 4, 126; here called E]. This new "confluence" theorem is analogous to Theorem 5 of E, with somewhat stronger conditions but a simpler proof. The intended application is (as in E) to the proof of the Church-Rosser Theorem in Church's λ -calculus. To deal with the new conditions a modified interpretation of a certain relation J is necessary (omission of condition (ii) of E, top of p. 241); but it seems to the reviewer (and

the author agrees) that this causes condition J_2' (replacing J_2 of E) to be no longer satisfied, and the proof of the Church-Rosser Theorem therefore fails.

M. H. A. Newman (Manchester).

Surányi, János. *Contributions to the reduction theory of the decision problem. II. Three universal, one existential quantifiers.* *Acta Math. Acad. Sci. Hungar.* 1, 261-271 (1950). (English. Russian summary)

The author proves that corresponding to any formula of the first order predicate calculus, there exists an equivalent binary formula of the form $(x)(y)(Ez)N_1 \& (x)(y)(u)N_2$, which can be transformed to either of the prenex forms $(x)(y)(u)(Ez)N$ and $(x)(y)(Ez)(u)N$, where $N = N_1 \& N_2$, and N_1 and N_2 do not contain quantifiers. This is a simultaneous improvement of results of Gödel and Pepis, which it contains as special cases. Its significance is that the number of universal and existential quantifiers is simultaneously restricted, which was not the case in previous results.

O. Frink (State College, Pa.).

Kalmár, László. *Contributions to the reduction theory of the decision problem. III. Prefix*

$$(x_1)(Ex_2) \cdots (Ex_{n-2})(x_{n-1})(x_n),$$

a single binary predicate. *Acta Math. Acad. Sci. Hungar.* 2, 19-38 (1951). (English. Russian summary)

The author shows that corresponding to any formula of the first order predicate calculus there may be constructed an equivalent formula, with matrix containing only a single predicate variable, and with prefix of the type appearing in the title of this paper.

O. Frink (State College, Pa.).

Mostowski, Andrzej. *Correction to the paper "Some impredicative definitions in the axiomatic set-theory".* *Fund. Math.* 38, 238 (1951).

See *Fund. Math.* 37, 111-124 (1950); these *Rev.* 12, 791.

*Robinson, Abraham. *On the metamathematics of algebra.* *Studies in Logic and the Foundations of Mathematics.* North-Holland Publishing Co., Amsterdam, 1951. ix+195 pp. 18.00 florins.

Die moderne Logik ist aus einer Kritik der Methoden der traditionellen Mathematik entstanden. Da aber diese Kritik immer noch nicht zu einem endgültigen Resultat gekommen ist, ist es verständlich, dass die traditionellen Methoden unbekümmert weiterexistieren, ja sogar die Kalküle der modernen Logik nur als neuen Gegenstand für ihre Anwendung betrachten. Verf. weist in dem vorliegenden Buch, das eine unwesentliche Modifikation seiner "thesis" von 1949 ist, nach, dass durch diese Betrachtungsweise die traditionelle Mathematik, insbesondere die Algebra, zu neuen Resultaten gelangen kann. Zunächst wird in Kap. I-III das Vollständigkeitstheorem für den elementaren Prädikatenkalkül (d. h. 1. Stufe) bewiesen, wobei Individuen und

Relationen in beliebiger Mächtigkeit vorhanden sein können. Für die Logik wichtig ist noch das spätere Resultat, dass jedes unendliche Modell eines Axiomensystems mit Identität ein echtes Obermodell besitzt. In Kap. IV, V wird der Vollständigkeitssatz benutzt, um Sätze wie: Jeder elementare (d. h. im elementaren Prädikatenkalkül formulierbare) Satz für kommutative Körper der Charakteristik 0 gilt auch für alle kommutativen Körper genügend hoher Charakteristik. Die Methode des Verf. führt auch zu einfachen Existenzbeweisen für nicht-archimedische, sowie für algebraisch-abgeschlossene Körper.

Die restlichen Kap. VI–XI verallgemeinern algebraische Begriffe wie Polynom, separabel und Ideal auf beliebige elementare Axiomensysteme K mit Identität. Jedes Prädikat $R(x_1, \dots, x_n; y)$ für das überall die eindeutige Existenz von y aus K ableitbar ist, heisst ein Prädikatenpolynom (prepolynomial). Unter ihnen lassen sich die "Polynome" definieren, die für kommutative Ringe mit den Polynomen im algebraischen Sinne übereinstimmen. Die algebraischen Gleichungen lassen sich verallgemeinern als gewisse Prädikate $q(x)$, für die aus K ableitbar ist, dass ihnen höchstens endlich viele x genügen. Auch hier lassen sich noch "algebraisch-abgeschlossene" Erweiterungen konstruieren. Eine interessante Verallgemeinerung bilden die metamathematisch definierten Ideale. Ist I_0 die Menge aller primitiven Aussagen über ein Modell M eines Axiomensystems K , und ist M' ein zu M homomorphes Modell, so wird die Menge BCI_0 der Aussagen $a = b$, für die $a' = b'$ in M' gilt, betrachtet. Die Aussagen, die aus $K \cup B$ und den primitiven Sätzen über M ableitbar sind, bilden ein Ideal. Für beliebiges BCI_0 entstehen die allgemeinen Ideale. Sie bilden einen distributiven Verband, so dass ein grosser Teil der algebraischen Idealtheorie übertragbar wird. Die metamathematische

Methode führt aber auch zu einem neuen—hinterher auch rein algebraisch definierbaren—Begriff, dem des "disjunktiven" Ideals, dessen Theorie entwickelt wird.

P. Lorenzen (Bonn).

✓*Robinson, Abraham. On the application of symbolic logic to algebra. Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 686–694. Amer. Math. Soc., Providence, R. I., 1952.

Ein zusammenfassender Bericht über die Möglichkeit der Anwendung nicht-finitar metamathematischer Methoden auf algebraische Probleme, die Verf. in seinem oben referierten Buche ausführlich entwickelt hat.

P. Lorenzen.

Reichenbach, Hans. Über die erkenntnistheoretische Problemlage und den Gebrauch einer dreiwertigen Logik in der Quantenmechanik. Z. Naturforschung 6a, 569–575 (1951).

The author argues that a three-valued logic, with truth values true, false, and undetermined, is necessary, or at least convenient and desirable, for a complete description of the facts of quantum mechanics. He cites well-known paradoxes and anomalies to show that in quantum mechanics, unlike classical mechanics, none of the distinct but equivalent descriptions which will account for "interphenomena" obey the principle of causality. By interphenomena he means what takes place between observations. Among his illustrations are the wave-corpuscle dualism and the interpretation of the positron as an electron running backwards in time. The failure of causality is connected with the need for a third truth value. Objections of Bohr, Born, and Pauli to the use of three-valued logic in quantum mechanics are answered.

O. Frink (State College, Pa.).

ALGEBRA

Touchard, Jacques. Sur un problème de configurations et sur les fractions continues. Canadian J. Math. 4, 2–25 (1952).

Démonstration et développement des résultats déjà publiés par l'auteur [C. R. Acad. Sci. Paris 230, 1997–1998 (1950); ces Rev. 12, 44] et se rattachant à l'étude du problème des timbres-poste [Canadian J. Math. 2, 385–398 (1950); ces Rev. 12, 312]. On dit que S' est un sous-système de S s'il est recouvert par un arc au moins de S et s'il n'est coupé par aucun arc de $S - S'$. Un système est propre s'il ne contient aucun sous-système. Une configuration est composée d'un système unique s'il est impossible de la séparer en deux parties n'ayant aucun point commun. Une configuration avec p points doubles étant représentée par x^p , l'ensemble de toutes les configurations de n arcs est représenté par le polynôme $T_n(x)$. L'ensemble des systèmes uniques est représenté par $S_n(x)$ et ceux qui sont propres par $P_n(x)$. Règles de formation des systèmes uniques. En posant: $a_p = (x^p - 1)/(x - 1)$ et $S_n(x) = R_n(a) = \sum C(n, i) a_i$, $i = 2, 3, \dots, n$, on peut construire par récurrence les $C(n, i)$. L'expression générale de $R_n(a)$ est

$$R_n(a) = a_1 a_2 \sum_{i_1=0}^n \sum_{i_2=0}^{n-i_1} \sum_{i_3=0}^{n-i_1-i_2} \dots \sum_{i_{n-2}=0}^{n-i_1-i_2-\dots-i_{n-3}} a_{i_1} a_{i_2} \dots a_{i_{n-2}}.$$

Les polynômes $S_n(x)$, $n \geq 2$, satisfont aux égalités remarquables:

$$S(-1) = 0; \quad S_n(j) = (-1)^{n-1} j^{2n-2}; \\ S_n(i) = (1+i)(1+2i)^{n-2}; \quad j^2 = -i^2 = 1.$$

Construction des S_n au moyen de fractions continues.

$$S_n(x) = g_n(x) - \sum g_{n-i} S_i, \quad i = 1, 2, \dots, n-1;$$

$$g_n(x) = -\sum (-1)^i D_{2n-2i}^{(n+1-i)(n-i)/2} (1-x)^{-n}; \\ i = 0, 1, 2, \dots, n$$

où les D sont les nombres de Segner-Delannoy. L'expression des T est ensuite donnée par: $T_n(x) = R_{n+1}(1, a_1, a_2, \dots, a_n)$, cette dernière fonction se déduisant de $R_{n+1}(a)$ en y remplaçant a_1 par 1, a_2 par a_1 , \dots , a_i par a_{i-1} . La détermination de $P_n(x)$ est réalisée en établissant la relation qui existe entre les fonctions génératrices de P et de S . Il en résulte, au moyen de notations convenables, une réciprocity entre les S et les P . Applications diverses. Tables de S et de g pour $n = 1$ à 6 et de P pour $n = 1$ à 5.

A. Sade.

Rodeja F., E. G.-. Note on determinants of sines and cosines. Proc. Amer. Math. Soc. 3, 198–205 (1952).

Let $S(k_1, k_2, \dots, k_n) = |\sin k_j \theta_i|$ be a determinant in the $2n$ variables $\theta_1, \theta_2, \dots, \theta_n, k_1, k_2, \dots, k_n$. It is proved that if $1 \leq k_1 < k_2 < \dots < k_n$ are all integers, then for all real $\theta_j, j = 1, 2, \dots, n$,

$$\frac{|S(k_1, k_2, \dots, k_n)|}{|S(1, 2, \dots, n)|} \leq \frac{\prod_{j=1}^n k_j \prod_{k>j=1}^n (k_k^2 - k_j^2)}{1!3!5! \dots (2n-1)!}$$

with equality holding in the limit as all $\theta_j \rightarrow 0$. A similar result is proved for determinants of cosines. These two theorems give the complete generalization of inequalities

proved by the reviewer [Trans. Amer. Math. Soc. 63, 175-192 (1948); these Rev. 9, 421].
A. W. Goodman.

Leavitt, W. G. Mappings of vector spaces and the theory of matrices. Amer. Math. Monthly 59, 219-222 (1952).

Weisel, Heinrich. Die Auflösung algebraischer Gleichungen in formaler Übereinstimmung. Math.-Phys. Semesterber. 2, 199-206 (1952).

Amante, Salvatore. Risoluzione dei sistemi lineari di equazioni fra matrici. Matematiche, Catania 6, 119-125 (1951).

The analogues of the classical theorems of Cramer and Rouché-Capelli are established for a system of linear matrix equations of the form $\sum_j A_{ij} X_j = K_i$, A_{ij} , X_j and K_i being square matrices of the same order. There is a curious misprint on p. 121 where "il reciproco dell'elemento a_{uu} di A " should read "l'elemento del reciproco di A ".

D. E. Rutherford (St. Andrews).

Fréchet, Maurice. Solutions non commutables de l'équation matricielle $e^{A+B} = e^A e^B$. C. R. Acad. Sci. Paris 233, 1339-1340 (1951).

The following result is announced: For non-commutative real or complex matrices A and B of order two to satisfy $e^A \cdot e^B = e^{A+B}$, it is necessary and sufficient that they satisfy relations of the type $e^A = pI$, $e^B = qI$ and $e^{A+B} = rI$.

W. Givens (Knoxville, Tenn.).

Ishaq, M. A note on Hadamard matrices. Ganita 1, 13-15 (1950).

An Hadamard matrix of order n is a matrix whose elements are ± 1 and whose determinant equals $n^{n/2}$. The author proves that each characteristic root λ of such a matrix has the absolute value $n^{1/2}$. Moreover he proves that with λ also n/λ is a characteristic root. But this is trivial since λ and n/λ are conjugate complex numbers.
A. Brauer.

Stein, P. A note on inequalities for the norm of a matrix. Amer. Math. Monthly 58, 558-559 (1951).

Let $A = (a_{ij})$ be a square matrix with real elements, D its determinant, $\text{adj } A$ its adjoint, and

$$N^2(A) = \text{trace } A'A = \sum_{i,j} a_{ij}^2$$

the square of its norm. Denote by m and M the smallest and the largest of the characteristic roots of the symmetric matrix $\frac{1}{2}(A+A')$, respectively. The author proves that

$$mN^2(\text{adj } A) \leq D \text{ trace } (\text{adj } A) \leq MN^2(\text{adj } A).$$

This gives for $D \neq 0$

$$mN^2(A^{-1}) \leq \text{trace } A^{-1} \leq MN^2(A^{-1}).$$

A similar result is obtained for matrices with complex elements.
A. Brauer (Chapel Hill, N. C.).

Gyires, B., und Varga, O. Anwendung von p -Vektoren auf derivierte Matrizen. Publ. Math. Debrecen 2, 137-145 (1951).

The authors consider n independent vectors,

$$a_i \quad (i, k = 1, \dots, n),$$

in an n -dimensional Euclidean space E_n . The totality of $N = C_p^n$, p -vectors which can be constructed from the vectors a_i determine the p th derived matrix. The authors'

purpose is to apply the geometry of p -vectors to the study of the p th derived matrix. Hence, the well known properties of p -vectors (content, the p -parallel-surfaces determined by a p -vector, the angle between two p -vectors) are discussed. It is shown that the p th derived matrix determines the linear manifold of vectors, a_i (that is, the original matrix). Further,

a converse of a theorem of E. Pascal [Repertorium der höheren Mathematik, v. I, 2nd ed., Teubner, Leipzig, 1910, p. 139] is proved. This theorem is that orthogonality of a_i

implies orthogonality of the p -vectors which constitute the p th derived matrix and conversely.
N. Coburn.

Brenner, J. The problem of unitary equivalence. Acta Math. 86, 297-308 (1951).

This paper concerns the condition under which two square matrices A_1, B_1 are equivalent under the unitary group, i.e., such that a square matrix X can be found for which $XA_1X^* = B_1$, $XX^* = I$. A canonical form $C_1 = XA_1X^*$ is found such that A_1 and B_1 are equivalent if and only if they have the same canonical form. Schur has shown that any matrix can be transformed by a unitary matrix into a triangular matrix, but it seems unwise to build on this, for the conditions are not easily obtained for transforming a triangular matrix into another triangular matrix with the same leading diagonal. The author, therefore suggests the following procedure.

Given a matrix A_1 , then $F_1 = A_1A_1^*$ is Hermitian, and when A_1 is transformed by a unitary matrix, F_1 is transformed by the same matrix. As a first step, then, F_1 is transformed into a diagonal matrix. Further transformations are then allowable by matrices which commute with the diagonal matrix F_1 . If the latent roots of F_1 are all distinct, a further transforming matrix must be of the form $\text{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots)$. This can be chosen to make every element in the first row of A_1 real and positive. If all elements in the first row of A_1 are non-zero, this gives the canonical form. If they are zero elements in the first row, then a further group of transformations is allowable which can be used to simplify subsequent rows. In the case of a set of equal latent roots of F_1 the corresponding square submatrix of A_1 can be transformed to diagonal form. The group of unitary matrices is next considered which commutes with F_1 and leaves invariant this diagonal submatrix, and such further simplifications are made as this allows. By an inductive method a canonical form is obtained for A_1 for all contingencies. The procedure and the canonical form may, however, be extremely complicated, and it seems likely that a simpler solution of the problem could be found.

D. E. Littlewood (Bangor).

Taussky, Olga. Classes of matrices and quadratic fields. II. J. London Math. Soc. 27, 237-239 (1952).

A new and shorter proof is given of the main theorem of a previous paper [Pacific J. Math. 1, 127-132 (1951); these Rev. 13, 201]. It is also proved that in the quadratic field over the rationals generated by \sqrt{m} , where m is square free, $m \equiv 2 \pmod{4}$ and the fundamental unit has norm $+1$, a matrix class corresponding to an ideal class of order 2 can contain a symmetric matrix only if its ideal class does not contain an invariant ideal.
W. Givens.

Ueno, Masato. On the normalization of bi-quadratic form. Kodai Math. Sem. Rep. 1951, 45-48 (1951).

A bi-quadratic form $f(x, y) = \sum_{i,j,k,l=1}^n C_{ijkl} x_i x_j y_k y_l$ can be reduced by independent orthogonal substitutions on the two

sets of variables to $\sum_{i,j=1}^n a_{ij}(x_i')^2(y_j')^2$ if and only if 1) the matrices $C_{ij} = \|C_{ij\alpha\beta}\|$ commute, and, 2) the $C'_{\alpha\beta} = \|C_{ij\alpha\beta}\|$ commute, where $C_{ij\alpha\beta} = C_{ji\alpha\beta} = C_{ij\beta\alpha}$. *W. Givens.*

Makar, Raouf H. Algebraic and non-algebraic infinite matrices. *Nederl. Akad. Wetensch. Proc. Ser. A.* **54** = *Indagationes Math.* **13**, 426-435 (1951).

The author calls an infinite matrix $A = (a_{ij})$, $i, j = 1, 2, \dots$ of complex numbers algebraic if the powers A^k , $k = 2, 3, \dots$ exist (in the sense of absolute convergence of all series involved) and do not depend on the order of multiplication and if $P(A) = 0$, where P is a polynomial. A is an essential root of P if $P(A) = 0$ and P is the polynomial of the smallest possible degree r ; this r is the order of A . Some of the results based on these definitions are: Every polynomial has a matrix as an essential root; all essential roots of a given polynomial are of the same order and have the same minimal equation; the order n of $P(A)$, where P is of degree r and A of order m , satisfies the inequality $m \leq n \leq mr$.

G. G. Lorents (Toronto, Ont.).

Cugiani, M. Risultante e teorema di Bézout. *Period. Mat.* (4) **30**, 12-32 (1952).

Abstract Algebra

★ **Birkhoff, Garrett.** Some problems of lattice theory. *Proceedings of the International Congress of Mathematicians*, Cambridge, Mass., 1950, vol. 2, pp. 4-7. *Amer. Math. Soc.*, Providence, R. I., 1952.

The revised edition of the author's *Lattice Theory* [*Amer. Math. Soc. Colloq. Publ.*, v. 25, New York, 1948; these Rev. **10**, 673] contains 111 unsolved problems. A list is first given here of 21 of these for which solutions have already been published or reported. Certain of the remaining unsolved problems, and other new ones suggested by them are then discussed. Attention is called to two important but difficult problems: to determine whether every finite lattice is isomorphic with a sublattice of the lattice of all partitions of a finite set; and to solve the decision problem for the free modular lattice with four generators. Interesting problems and results concerning the lattice of all subgroups of a group and the lattice of all subalgebras of a Lie algebra are mentioned; in particular the question of when these lattices are complemented or relatively complemented arises.

The author then discusses problems connected with applications of lattice theory, in particular applications of the theory of lattice-ordered groups and semigroups. There is need also for a general theory of lattice-ordered rings. The author's decomposition theory for averaging operators is an example in this direction. Problems connected with the possible existence of a natural simple ordering of the ring of real functions of a real variable are pointed out. Finally a generalized lattice formulation of Duhamel's principle is described.

O. Frink (State College, Pa.).

Miller, D. G. Postulates for Boolean algebra. *Amer. Math. Monthly* **59**, 93-96 (1952).

The author gives a set of seven postulates for Boolean algebra in terms of multiplication, ring addition (or symmetric difference) and the unit element 1. The first three are postulates of existence and closure, while the four formal postulates are: $a1 = a$, $a + (b + b) = a$, $[a(bb)]c = (cb)a$, and $a[(b + c) + d] = a(d + c) + ab$. These are shown to be equivalent

to a standard set due to B. A. Bernstein. They are also proved independent by means of independence examples. The author also points out that if the first formal postulate is strengthened to $a1 = 1a = a$, then the last two may be replaced by the single complicated postulate

$$([a(bb)]c)[(d + e) + f] = [(cb)a](f + e) + [(cb)a]d.$$

O. Frink (State College, Pa.).

Wooyenaka, Yuki. Remark on a set of postulates for distributive lattices. *Proc. Japan Acad.* **27**, 162-165 (1951).

Independent sets of postulates for distributive lattices are given, which in particular solve Problem 65 of the reviewer's "Lattice Theory", 2d ed. [*Amer. Math. Soc. Colloq. Publ.* v. 25, New York, 1948; these Rev. **10**, 673]. It is noted that, in the postulates given on p. 135 of "Lattice Theory", it should be added that the I in (2) is the I of (3). (This was explicitly assumed in the original derivation of these postulates, by G. D. Birkhoff and the reviewer [*Trans. Amer. Math. Soc.* **60**, 3-11 (1946); these Rev. **8**, 192]. See also R. Croiset, *Canadian J. Math.* **3**, 24-77 (1951); these Rev. **12**, 472.) *G. Birkhoff (Cambridge, Mass.).*

Lesieur, Léonce. Conditions suffisantes pour que, dans un treillis multiplicatif complet, la condition de chaîne descendante entraîne la condition de chaîne ascendante. *C. R. Acad. Sci. Paris* **234**, 1017-1019 (1952).

The conditions are that L be a complete multiplicative modular "intercontinuum" lattice whose O and I are the multiplicative zero and unit, and that the mapping $x \rightarrow xa$ be the union of principal endomorphisms and also the union of essential endomorphisms (η is essential if $x\eta \cup h = y\eta \cup h$ implies the existence of x' and y' with $x \cup x' = y \cup y'$, $x'\eta \leq h$, $y'\eta \leq h$). *P. M. Whitman (Silver Spring, Md.).*

Croiset, Robert. Contribution à l'étude des treillis semi-modulaires de longueur infinie. *Ann. Sci. École Norm. Sup.* (3) **68**, 203-265 (1951).

The author studies 23 conditions for "semi-modularity" which are equivalent in lattices of finite length, but no two of which are equivalent in general lattices. He determines all implication relations between these conditions: in general lattices, in lattices satisfying the ascending chain condition (in which case there are 15 equivalence classes), and in lattices satisfying the descending chain condition (in which case 10 of the conditions are in one equivalence class, and the remaining 13 mostly inequivalent). The self-dual lattice of all closed subspaces of Hilbert space satisfies most of the conditions which are based on "covering" relations, but only a few of the other conditions for "semi-modularity". Many of the results had already been announced by the author [*C. R. Acad. Sci. Paris* **231**, 12-14, 1399-1401 (1950); these Rev. **12**, 4, 473]. *G. Birkhoff (Cambridge, Mass.).*

Novotný, Miroslav. Les systèmes à deux compositions avec une loi distributive. *Publ. Fac. Sci. Univ. Masaryk* no. 321, 49-68 (1951). (French. Russian summary)

This paper deals with algebraic systems which are closed under the two operations of addition and multiplication and which satisfy the left distributive law $a(b + c) = ab + ac$. The fact that multiplication is then an endomorphism of the additive groupoid is used to discuss the problem of constructing such systems on a given additive groupoid. The results for the most part are either well known or are obvious generalizations from ordinary ring theory.

D. C. Murdoch (Vancouver, B. C.).

Ikeda, Masatoshi. Some generalizations of quasi-Frobenius rings. Osaka Math. J. 3, 227-239 (1951).

In a ring A with minimum condition, let $l(S)$ and $r(S)$ be the set of elements annihilating a set S respectively on the left and on the right. A is characterized in terms of conditions 1) $l(r(a_i)) = a_i$ for left ideals and 2) $r(l(a_i)) = a_i$ for right ideals. If 1) holds for all left ideals we call A a D_1 -ring, but if 1) is assumed to hold for simple nilpotent ideals we call A an $S\cdot D_1$ -ring. Similar definitions are made from condition 2). If A is both a D_1 and a D_r ring then A is a quasi-Frobenius ring. It is shown that any one of these conditions implies that A has a unit. An $S\cdot D_1$ -ring is characterized in terms of a set of primitive orthogonal idempotents and their left ideals, and further results of this type are obtained. Finally A is said to be uni-serial if every indecomposable left ideal Ae has a unique composition series $Ae \supset Ne \supset N^2e \supset \dots$ where N is the radical of A . It is shown that every residue class ring of A is quasi-Frobenius if and only if A is uni-serial.

Marshall Hall (Washington, D. C.).

Jacobson, N., and Rickart, C. E. Homomorphisms of Jordan rings of self-adjoint elements. Trans. Amer. Math. Soc. 72, 310-322 (1952).

Soit S un anneau ayant un élément unité, $A = S_n$ l'anneau des matrices d'ordre n sur A , e_{ij} les unités matricielles formant une base de A sur S . Les auteurs disent qu'une involution (antiautomorphisme involutif) $x \rightarrow x^*$ de A est canonique si $e_{ii}^* = e_{ii}$: on a alors, pour $x = \sum_{i,j} x_{ij} e_{ij}$, $x^* = \sum_{i,j} x_{ji}^{-1} e_{ji}$ où $\alpha \rightarrow \alpha^{-1}$ est une involution dans S et les γ_i sont invariants par cette involution et inversibles. Soit H l'anneau de Jordan formé des éléments de A invariants par une involution canonique; on suppose en outre que tout élément de H est une "trace", c'est-à-dire de la forme $a + a^*$ (ce qui est toujours vérifié si on peut diviser par 2 dans A); alors, si $n \geq 3$, H est simple si A est simple. Le théorème principal du mémoire affirme que, dans les conditions précédentes (A non nécessairement simple), tout homomorphisme de Jordan $x \rightarrow x^j$ de H sur un anneau de Jordan spécial peut être étendu d'une seule manière à un homomorphisme (associatif) de A sur l'anneau associatif enveloppe de l'image de H . Ils montrent en outre comment ce résultat peut se généraliser aux anneaux de Jordan définis de la même manière que H , mais à partir d'un anneau simple infini possédant des idéaux minimaux ou à partir d'un anneau primitif possédant des idéaux minimaux. J. Dieudonné.

Atsuchi, Masahiko. Remark on the structure of Lie and Jordan rings. J. Fac. Sci. Hokkaido Univ. Ser. I. 11, 126-128 (1950).

Let L be a module with operator satisfying

- (1) $a \times (b+c) = a \times b + a \times c$, $(b+c) = b \times a + c \times a$
- (2) $\lambda(a \times b) = (\lambda a) \times b = a \times (\lambda b)$,

where $a, b, c \in L$, and $\lambda \in P$, the operator field. Let ϕ be an operator homomorphism of L into the non-commutative ring S , and assume $\phi(a \times b)$ is a polynomial in $\phi(a)$, $\phi(b)$ of degree less than the characteristic of P . Then

$$\phi(a \times b) = \xi \phi(a) \phi(b) + \xi' \phi(a) \phi(b), \quad \xi, \xi' \in P.$$

If L satisfies (3) $a \times b = b \times a$, then $\xi = \xi'$; if L satisfies (3') $a \times a = 0$ and $S^2 \neq \{0\}$, then $\xi' = -\xi$. If $S^2 = \{0\}$, then $\phi(a \times b) = \lambda xy$ and $\phi(L)$ is an associative ring.

G. D. Mostow (Syracuse, N. Y.).

Kuročkin, V. M. Correction to the paper, "The representation of Lie rings by associative rings." Mat. Sbornik N.S. 30(72), 463 (1952). (Russian)

See Mat. Sbornik N.S. 28(70), 467-472 (1951); these Rev. 12, 799.

Kaplansky, Irving. Modules over Dedekind rings and valuation rings. Trans. Amer. Math. Soc. 72, 327-340 (1952).

The first objective of this paper is "to push forward the theory of modules over Dedekind rings to approximately the same point as the known theory of modules over principal ideal rings". If M is a module over an integral domain R , the set of elements in M with nonzero order ideals forms the torsion submodule T . The quotient module M/T is torsion-free; that is, all nonzero elements of M/T have zero order. Let R be a Dedekind ring, a domain in which every ideal is invertible. Then every finitely generated R -module is the direct sum of its torsion submodule and modules of rank one; the latter, being torsion-free, are isomorphic to invertible ideals in R . (This result is actually proved for domains R in which every finitely generated ideal is invertible.) The structure of a torsion R -module may be obtained by expressing it as the direct sum of primary modules, and considering each p -primary module as a module over R_p , the quotient ring of R relative to the prime ideal p . Every direct summand S of an R -module M is "pure" in the following sense: $\alpha S = S \cap \alpha M$ for every α in R . The author shows that conversely a pure submodule S of a module M over a Dedekind ring is a direct summand if either of the following requirements is met: (i) M/S is the direct sum of cyclic modules and finitely generated torsion-free modules of rank one; or (ii) S is of bounded order; i.e., $\gamma S = 0$ for some nonzero γ in R . A module M is divisible if $\alpha M = M$ for all nonzero α in R . It is shown that any R -module M may be written uniquely as a direct sum: $M = D \oplus E$, where D is divisible, and E has no divisible submodules. A complete structure theorem for divisible modules is obtained. Finally it is shown that an indecomposable R -module must be either torsion-free or torsion; and, in the latter case, the possible types are specified.

The second portion of this paper concerns modules over valuation rings. If R is a maximal valuation ring, any torsion-free R -module of countable rank is a direct sum of modules of rank one. One way of describing a maximal valuation ring R (with quotient field K) is to say that it satisfies the following condition: whenever $\alpha, \epsilon \in K$ and I, ϵ (integral or fractional) ideals such that the congruences $x \equiv \alpha \pmod{I}$ can be solved in pairs, then there exists in K a simultaneous solution of all the congruences. If this condition holds whenever the intersection of the ideals I is nonzero, R is called almost-maximal. The author's final result is that any finitely generated module over an almost-maximal valuation ring is a direct sum of cyclic modules. B. N. Moyls.

Lazard, Michel. Sur les algèbres enveloppantes universelles de certaines algèbres de Lie. C. R. Acad. Sci. Paris 234, 788-791 (1952).

The author constructs a universal enveloping algebra A for a Lie algebra L over a ring, thus generalizing the Birkhoff-Witt construction. He defines a canonical mapping ω of A onto $\Sigma(L)$, the symmetric algebra of L , taking ideals into homogeneous ideals in $\Sigma(L)$ and strictly monotone on ideals; furthermore, x_1, \dots, x_n is a base for the ideal KCA if and only if $\omega(x_1), \dots, \omega(x_n)$ is a base for $\omega(K)$.

G. D. Mostow (Syracuse, N. Y.).

Whaples, George. A theorem on cyclic algebras. *Ann. of Math.* (2) **55**, 367-372 (1952).

It is proved that $(C_1L/L, \sigma_1^{1/4}, A_1) \sim (C_2L/L, \sigma_2^{1/4}, A_2)$ implies $(C_1/k, \sigma_1, N_{L/k}A_1) \sim (C_2/k, \sigma_2, N_{L/k}A_2)$, where C_1, C_2 are cyclic and L separable over an (arbitrary) field k , σ_i and $\sigma_i^{1/4}$ are generating automorphisms for C_i/k and C_iL/L respectively, and $A_i \in L$. This gives the algebraic background of the translation theorem for norm residue symbols. The earlier proofs of the last theorem depend on particular properties of p -adic number fields (but replace "implies" by "if and only if").

T. Nakayama (Nagoya).

Theory of Groups

Norton, D. A. Hamiltonian loops. *Proc. Amer. Math. Soc.* **3**, 56-65 (1952).

This paper contains a discussion of the structure of Hamiltonian loops, that is, loops in which every subloop is normal. The category of Hamiltonian loops is too broad a one to admit as precise a characterisation as is possible for Hamiltonian groups, since loops exist of every order except 4 which have no proper subloops and which are therefore trivially Hamiltonian. Moreover, it is stated that a Hamiltonian loop can be constructed with a prescribed composition series of subloops although the actual construction is omitted. Consequently, the problem is somewhat specialized by imposing associativity conditions of increasing severity on the Hamiltonian loop, and investigating its structure under each of these additional conditions. Sample results are: (1) A power associative Hamiltonian loop in which every element has finite order is the direct product of p -loops; (2) a commutative Hamiltonian diassociative loop (i.e. one in which any two elements generate a group) is an Abelian group; (3) a Hamiltonian diassociative loop, in which any three elements which associate in some order generate a group, is either (i) an Abelian group, or (ii) a Hamiltonian group, or (iii) the direct product of an Abelian group with exponent 2, an Abelian group with elements of odd order and a Cayley loop, a Cayley loop being the loop formed by the basis elements of a Cayley-Dickson algebra. Necessary and sufficient conditions that the direct product of two Hamiltonian loops be Hamiltonian are given.

D. C. Murdoch (Vancouver, B. C.).

Green, J. A., and Rees, D. On semi-groups in which $x^r = x$. *Proc. Cambridge Philos. Soc.* **48**, 35-40 (1952).

Denote by S_n the semigroup generated by n of its elements, in which every element x satisfies the equation $x^r = x$, the semigroup being otherwise free. Similarly, let $B_{n,r-1}$ be the group generated by n of its elements in which every element x satisfies the equation $x^{r-1} = 1$, the group being otherwise free. The Burnside conjecture states that $B_{n,r-1}$ is finite for all n and this is known to be true for $r=2, 3, 4$. It is shown in this paper that the proposition, " S_n is finite for all n ", is equivalent to the Burnside conjecture, and a formula is found for the order of S_n for the case $r=2$.

D. C. Murdoch (Vancouver, B. C.).

Mann, Henry B. On products of sets of group elements. *Canadian J. Math.* **4**, 64-66 (1952).

Let A_1, \dots, A_m be distinct elements of an abelian group \mathfrak{g} , and let B_1, \dots, B_n be distinct elements of \mathfrak{g} . (The author

omits "distinct", but it is obviously implied.) Let C_1, \dots, C_s be the distinct elements of \mathfrak{g} which are representable as $A_i B_j$. Supposing these not to exhaust \mathfrak{g} , let \bar{C} be any other element of \mathfrak{g} . The main theorem of the paper is that the set B_1, \dots, B_n can be enlarged to a set B_1, \dots, B_N , whereupon the set C_1, \dots, C_s is enlarged to C_1, \dots, C_N , in such a way that (1) the elements of \mathfrak{g} other than C_1, \dots, C_N constitute all elements $\mathfrak{H}\bar{C}$, where \mathfrak{H} is some subgroup of \mathfrak{g} ; (2) $S - s = N - n$. The proof is closely related to the author's proof of his theorem on the addition of sets of integers [*Ann. of Math.* **43**, 523-527 (1942); these *Rev.* **4**, 35]. As a corollary the author deduces the theorem of Cauchy concerning the addition of residue-classes to a prime modulus, and its extension by I. Chowla. (For references to literature see H. Davenport, *J. London Math. Soc.* **22**, 100-101 (1947); these *Rev.* **9**, 271.)

H. Davenport (London).

Szele, T. On a theorem of Pontrjagin. *Acta Math. Acad. Sci. Hungar.* **2**, 121-123 (1951). (English. Russian summary)

The author gives a new proof of the result [Pontrjagin, *Topological groups*, Princeton Univ. Press, 1939, pp. 168-169; these *Rev.* **1**, 44] that a countable torsion-free abelian group is the direct sum of cyclic groups if and only if every increasing sequence of subgroups of an arbitrary finite rank r contains only a finite number of different subgroups. He establishes an equivalent theorem: A countable torsion-free abelian group G is the direct sum of cyclic groups if and only if in G every subgroup of finite rank is finitely generated.

F. Haimo (St. Louis, Mo.).

Dieudonné, Jean. Sur les p -groupes abéliens infinis. *Portugaliae Math.* **11**, 1-5 (1952).

Let G be a primary abelian group, H a subgroup such that: (1) G/H is a direct sum of cyclic groups; (2) H is the union of an ascending sequence of subgroups of bounded height (relative to G). Then G is a direct sum of cyclic groups. The special case where $H=G$ is due to Kulikov [*Mat. Sbornik N.S.* **16**(58), 129-162 (1945); these *Rev.* **8**, 252], and the author's proof proceeds by a further exploitation of Kulikov's method. The paper concludes with some remarks and examples; in particular, an example shows that it is insufficient to assume that H is a direct sum of cyclic groups, even if G has no elements of infinite height.

I. Kaplansky (Chicago, Ill.).

Bays, S. Sur l'imprimitivité des groupes de substitutions par rapport aux i -uples. *Comment. Math. Helv.* **25**, 298-310 (1951).

Let H and H_G be the subgroups of an n -fold transitive substitution group G that leave fixed a particular arrangement A of i letters (called an i -uple A) or the corresponding combination C ($i \leq n$). To every subgroup K between H and G there corresponds a repartition of the i -uples (A) into sets of imprimitivity, determined by the sets of right cosets of H which belong in a particular coset of K in G . For $i > 1$ the subgroup H_G contains H properly, and there are certain necessary imprimitive repartitions, such as $(ab, ba), \dots$ for couples. Furthermore, to each subgroup K between H and G there correspond $i!$ conjugate imprimitive repartitions (not necessarily distinct). Primitivity of a group with respect to i -uples is defined to exclude all subgroups between H and G except those required by the necessary imprimitive repartitions.

J. S. Frame (East Lansing, Mich.).

Azleckij, S. P. On systems of Sylow classes of a finite group. *Mat. Sbornik N.S.* 29(71), 581-586 (1951). (Russian)

For terminology and notation, see the author's earlier paper [*Mat. Sbornik N.S.* 28(70), 461-466 (1951); these *Rev.* 12, 799]. Let Θ be the intersection of all maximal normal subgroups of G , let Σ be the set of those (P_i) not contained in Θ , let s be the number of these latter. Then Σ is a Sylow system; it contains every minimal Sylow system; a necessary and sufficient condition that Σ be itself a minimal system is that Θ contain G' . Let G be said to have property $(\#)$ provided it is generated by the collection of those (P_i) not contained in G' . If $s > 1$, a group of Sylow rank $s-1$ has $(\#)$. If $k > 2$, a group of Sylow rank $k-2$ with property $(*)$ has $(\#)$. A group of Sylow rank r has $(\#)$ if and only if the index of G' is divisible by r distinct primes. A direct product of groups each satisfying $(\#)$ satisfies $(\#)$. If $0 \leq k \leq k-1$, a group of Sylow rank $k-h$ is said to be of type $k-h$ provided every subgroup has Sylow rank not less than $k-h$, where k is the number of distinct primes dividing the order of the subgroup. A group of type k is special; a group of type $k-1$ is solvable. A group of type $k-2$ with $(*)$ in which every subgroup of Sylow rank $k-2$ satisfies $(*)$ is solvable. *R. A. Good* (College Park, Md.).

Knoche, Hans-Georg. Über den Frobenius'schen Klassenbegriff in nilpotenten Gruppen. *Math. Z.* 55, 71-83 (1951).

Let \mathcal{G} be any finite nilpotent group of class c and let \mathcal{Z}_i be any term in its ascending central series. The i th normalizer (Normalisator i -ter Stufe) of an element A of \mathcal{G} is the set $\mathcal{N}_i(A)$ of all elements X of \mathcal{G} whose commutators (X, A) belong to \mathcal{Z}_i . It follows that $\mathcal{N}_0(A) = \mathcal{N}(A)$, the ordinary normalizer of A . Moreover $\mathcal{N}_i(A)$ is a normal subgroup of $\mathcal{N}_{i+1}(A)$, and $\mathcal{N}_{c-1}(A) = \mathcal{G}$. In general, if A belongs to \mathcal{Z}_{i+1} but not to \mathcal{Z}_i , then $\mathcal{N}_i(A) = \mathcal{G}$ but $\mathcal{N}_{i-1}(A) \neq \mathcal{G}$. Corresponding to this generalized concept of normalizer there is a decomposition of \mathcal{G} into generalized classes of conjugate elements. An element A of \mathcal{G} is said to be i -stufig conjugate to an element B if $A = S^{-1}BS$ where S lies in $\mathcal{N}_i(A)$. This relation is reflexive, symmetric and transitive, and hence divides \mathcal{G} into disjoint classes of i -stufig conjugate elements. The classes of $(c-1)$ -stufig conjugate elements coincide with the ordinary classes of conjugate elements of Frobenius. The remainder of the paper deals with p -groups whose commutator groups have order p . Generating relations for such a group are given and several results concerning maximal normal subgroups are proved. *D. C. Murdoch*.

Sanov, I. N. Establishment of a connection between periodic groups with period a prime number and Lie rings. *Izvestiya Akad. Nauk SSSR. Ser. Mat.* 16, 23-58 (1952). (Russian)

Following Magnus [*Ann. of Math.* (2) 52, 111-126 (1950); these *Rev.* 12, 476] the author studies the Burnside groups of prime period p using the ring R of formal power series in associative indeterminates over the rational field. A Lie product $[xy]$ is defined in R by the rule $[xy] = xy - yx$. If x_1, \dots, x_n are generating elements of R , then $\exp(x_1), \dots, \exp(x_n)$ generate a free group F . If $F = F_1, F_2, \dots, F_i, \dots$ is the lower central series for F , then for $g \in F_i$, $g = \exp(u_i + u_{i+1} + \dots)$ with u_j a Lie polynomial of degree j . The mapping $g \rightarrow u_i$ gives an isomorphism between F_i/F_{i+1} and the additive group of the u_i . The basic formula is the expression for s as defined from $\exp(x) \exp(y) = \exp(z)$.

These representations are applied to the Burnside group B of period p which is the factor group of F modulo the normal subgroup generated by p th powers. The main results consist in describing in detail the terms of the lower central series of B up to B_{2p-2} . If A_i is the Lie subring corresponding to B_i/B_{i+1} , then $A_i = 0 \pmod{p}$ for $i = 1, \dots, p-1$, and for $i = p, \dots, 2p-2$ the A_i are the elements of the ideal J consisting of multiples of p and elements derivable from $[xy^{p-1}]$ by Hausdorff differentiation and substitutions. In particular it is shown for the Burnside group of period 5 and two generators that its order is at least 5^4 , the orders of the first eight factor groups of the lower central series being respectively $5^2, 5, 5^3, 5^3, 5^3, 5^4, 5^4$, and 5^4 .

Marshall Hall (Washington, D. C.).

Chen, K. T. Commutator calculus and link invariants. *Proc. Amer. Math. Soc.* 3, 44-55 (1952).

Suppose that G is a finitely generated group such that $G/[G, G]$ has a basis of n elements, and let G be presented by $n+k$ generators and $k+q$ relations. Then for each positive i , the author constructs a group H presented by n generators and q relations such that H/H_i is isomorphic with G/G_i , where H_i and G_i are the i th terms of the lower central series of G and H respectively [$H_1 = H, G_1 = G$]. Applications are given to the case G is the group of a link L , notably, a presentation of G/G_i is constructed in terms of the number of components of L and the linking numbers of the different pairs of components. *D. G. Higman*.

Takahasi, Mutuo. Note on chain conditions in free groups. *Osaka Math. J.* 3, 221-225 (1951).

It is known that the ascending chain condition holds in finitely generated free groups, but not the descending chain condition. In infinitely generated free groups, neither holds. With additional restrictions the author proves various chain conditions for free groups. Let F be any free group and H a finitely generated subgroup. Then he shows that there are only a finite number of subgroups U containing H which do not have free factors containing H . Applying this result he finds that if $H_1 \supseteq H_2 \supseteq \dots \supseteq H_n \dots$ is a descending chain in a free group F and the intersection of all H_i is not the identity, then all H_n for n greater than some N have a common free factor K . This generalizes a theorem of the reviewer's and leads easily to the theorem of Magnus: the intersection of the higher commutator subgroups of a free group is the identity. *Marshall Hall*.

Neumann, B. H. A note on algebraically closed groups. *J. London Math. Soc.* 27, 247-249 (1952).

W. R. Scott has introduced definitions of algebraically closed and weakly algebraically closed groups [*Proc. Amer. Math. Soc.* 2, 118-121 (1951); these *Rev.* 12, 671]. An algebraically closed group is weakly algebraically closed. The present author proves that conversely every weakly algebraically closed group $\neq 1$ is algebraically closed, and furthermore that every algebraically closed group is simple. *D. G. Higman* (Urbana, Ill.).

Scott, W. R. Groups and cardinal numbers. *Amer. J. Math.* 74, 187-197 (1952).

Let G be an infinite group. For $a \in G$, let $E(a)$ be the set of all elements $g \in G$ such that a is non- ϵ [g] where $\{g\}$ is the cyclic group generated by g . Let K denote the set of all elements $g \in G$ such that $E(g)$ has a cardinal number less than the order of G . Let D be the intersection of all subgroups of G of the same order as G . The author proves that K is either

a cyclic group of prime power order or a group of type p^* , in both cases contained in the center of G . Moreover, $K=1$ if G is not periodic. On the other hand, K is a group of type p^* if and only if G has a p^* subgroup C of finite index contained in the center of G . If such a C exists, then $C=K=D$. For a non-countable G , K is a cyclic group of prime power order.

The problem of determining K and D is completely solved for abelian groups G . It is shown that in this case always $K=D$, and K is a group of type p^* or $K=1$, according as $G=H \times F$ with a p^* group H and a finite group F holds or not. As an especially remarkable application of the results of this paper, the following should be mentioned: if G is an arbitrary abelian group every proper subgroup of which is of order less than that of G , then G is either a finite group or a group of type p^* . *T. Szec (Debrecen).*

Deheuvels, René. Relations entre systèmes de groupes. Applications à la théorie des faisceaux. Bull. Tech. Univ. Istanbul 3 (1950), no. 1, 1-21 (1951). (French. Turkish summary)

This paper studies comparisons of direct and inverse systems of groups indexed by filters of open or closed subsets of a topological space E . For closed X , $Y \subseteq E$ let g_X be a group, written additively, and let h^{XY} be a homomorphism of g_X into g_Y , where $X \supset Y$. If the group attached to the empty set is zero, and if the set of groups and homomorphisms forms a direct system [Lefschetz, Algebraic topology, Amer. Math. Soc. Colloq. Publ., v. 27, New York, 1942, Chap. 1; these Rev. 4, 84], the whole system is called a net \mathfrak{F} of groups on E . Attached to a closed set X are filters of subsets of E , for example, $f_X(V)$, the set of closed neighborhoods of X , and $f_X(v)$, the set of closed neighborhoods of X with complements contained in compact sets. A net \mathfrak{F} is called continuous (strongly continuous) if, for each closed X , g_X is the direct limit of the system of those groups of the net indexed by $f_X(V)$ ($f_X(v)$). The theorems of the last half of the paper study relations between the properties, such as normality, of the space, the continuity properties of the net, and the structure of the set of subsets X such that for given A and fixed non-zero a in g_A either $X \subset A$ and $h^{AX}a$ is not zero or $X \supset A$ and there exists x in g_X such that $h^{XA}x$ is a . The first half of the paper considers comparison of abstract directed systems and then of direct and inverse systems of groups. It is shown that if f is comparable to f' , then the comparison homomorphisms determine a homomorphism of the limit groups of f and f' . *M. M. Day.*

Kac, G. I. Characters of the representations of the unimodular group. Doklady Akad. Nauk SSSR (N.S.) 83, 9-12 (1952). (Russian)

The author derives explicit expressions for the characters of the irreducible finite-dimensional representations of the complex unimodular group, by algebraic means.

I. E. Segal (Chicago, Ill.).

Ganea, Tudor. Du prolongement des représentations locales des groupes topologiques. Acta Sci. Math. Szeged 14, 115-124 (1951).

This paper is concerned with extensions of the following theorem of Schreier: Let φ be a local representation, in an abstract group H , of a connected neighborhood of the identity of a connected, locally connected, and simply connected topological group G . Then there exists a unique extension of φ to all of G . The chief result is: Let G be a topo-

logical group which is generated and locally generated by each neighborhood of the identity. In order that every local representation of G can be extended it is necessary and sufficient that every covering of G be degenerate. Examples are given to show that the conditions in the first-stated theorem are more restrictive than the conditions in the second theorem. *D. Montgomery (Princeton, N. J.).*

Garnir, H. G. Théorie de la représentation linéaire des groupes alternés. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° (2) 26, no. 1615, 22 pp. (1951).

Frobenius' theory of irreducible characters of alternative groups and the author's previous study [Mém. Soc. Roy. Soc. Liège (4) 10, no. 2 (1950); these Rev. 12, 77] of Young's semi-normal representations of symmetric groups are combined to construct irreducible representations of alternative groups, re-establishing in particular Thrall's result [Duke Math. J. 8, 611-624 (1941); these Rev. 3, 195]. The representations of A_4 and A_5 are actually constructed.

T. Nakayama (Nagoya).

Itô, Seizô. Unitary representations of some linear groups. Nagoya Math. J. 4, 1-13 (1952).

Let G be the group of sense-preserving rigid motions in the plane. Inspired by Gelfand and Neumark's treatment of the group of linear transformations of the line, the author determines explicitly the irreducible unitary representations of G , the cyclic representations of G and the accompanying positive definite functions. He overlooks the fact that the irreducible representations of G have also been discussed by Wigner [Ann. of Math. (2) 40, 149-204 (1939)] and that his theorem about them is a special case of a theorem of the reviewer about the representations of semi-direct products [Proc. Nat. Acad. Sci. U. S. A. 35, 537-545 (1949); these Rev. 11, 158]. *G. W. Mackey (Cambridge, Mass.).*

***Gel'fand, I. M., i Naimark, M. A. Unitarnye predstavleniya klassičeskikh grupp. [Unitary representations of the classical groups.]** Trudy Mat. Inst. Steklov., vol. 36, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1950. 288 pp. 16 rubles.

This is a fairly comprehensive account of the irreducible continuous unitary representations of the classical simple (complex) Lie groups, and of related aspects of harmonic analysis on such groups. It supersedes the material published in a long series of earlier notes. The results are for the most part very explicit and apparently well adapted to specific applications. There is however an in part concomitant lack of coordination with general Lie and topological group theory, and in particular, different series of groups are treated separately and the exceptional groups are not covered. The main results are the determination of all the irreducible (weakly) continuous unitary representations of the stated groups on a Hilbert space, the development of a theory of characters for such representations (which are all infinite-dimensional, so that the conventional definition of character can not be applied), and of an analog to the Plancherel formula (worked out only for the case of the unimodular group). In general the results parallel fairly closely the results obtained by the same authors for the case of the Lorentz group [Izvestiya Akad. Nauk SSSR. Ser. Mat. 11, 411-504 (1947); these Rev. 9, 495] (this being locally isomorphic to the complex simple Lie group of lowest dimension).

Actually the authors obtain not only unitary representations but a family of representations, by linear transformations on function spaces, depending on a finite set of parameters, which for one set of values of the parameters yields the unitary representations, but which for certain other values yields the finite-dimensional (necessarily continuous and non-unitary) irreducible representations first obtained by Cartan in 1914 in a form less explicit than the present one. For parameters that are of neither type the representation is not finite-dimensional and can not be made unitary by a suitable choice of inner product between the functions in the representation space. For example, if G is the 2×2 complex unimodular group, for each integer m and complex number ρ there is a representation T on the space of functions f of one complex variable z given by the equation

$$(T_g f)(z) = f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) |\beta z + \delta|^{-m-\rho-1} (\beta z + \delta)^{-m},$$

where

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

If the inner product $(f_1, f_2) = \int f_1(z) \overline{f_2(z)} dx dy$ ($z = x + iy$) is introduced and the representation space is limited to functions f for which (f, f) is finite, then a unitary representation is obtained on the corresponding Hilbert space, provided that ρ is pure imaginary. This is the "principal" series of unitary representations, and includes all the representations contained (in a reasonable sense) in the regular representation of G . It is noteworthy that the same formula for T_g can be used with quite a different inner product:

$$(f_1, f_2) = \int |z_1 - z_2|^{-2+\rho} f_1(z_1) \overline{f_2(z_2)} dx_1 dy_1 dx_2 dy_2 \quad (z_j = x_j + iy_j),$$

and for $m=0$ and $0 < \rho < 2$ the resulting representations are again unitary (and inequivalent to the preceding ones), constituting the "complementary" series of unitary representations. These are all the (non-trivial) irreducible unitary representations of G . The finite-dimensional irreducible representations are obtained when ρ is an integer of the same parity as m and f is restricted to have the form $f(z) = \sum \alpha_{ij} z^i \bar{z}^j$, where the α_{ij} are arbitrary complex numbers and $0 \leq i \leq \frac{1}{2}(-m+\rho-2)$, $0 \leq j \leq \frac{1}{2}(m+\rho-2)$, this class of functions being invariant under the T_g .

The general character of the results is well indicated by the case when G is the $n \times n$ complex unimodular group ($n > 1$). Almost all elements g of G (relative to Haar measure) can be uniquely represented in the form $g = ks$, where s is in the collection Z of all complex matrices whose diagonal elements are all one and whose superdiagonal elements vanish, while k is in the collection K of elements of G with vanishing subdiagonal elements. In the case $n=2$, Z is the set of all matrices of the form $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, so that it corresponds naturally to the complex plane, and in general the representation spaces for the irreducible unitary representations of G consist of functions on Z . The representations are in fact all (except certain "degenerate" ones) of the form $(T_g f)(z) = f(zg) \phi(z, g)$, where zg is that element s_1 such that $zg = ks_1$ with $k \in K$, and ϕ being a suitable fixed function. Specifically, if $\delta_1, \delta_2, \dots, \delta_n$ are the eigenvalues of k , then $\phi(z, g) = |\delta_1|^{m_1+\rho_1} |\delta_2|^{m_2+\rho_2} \dots$ where the m_j are integers. The m_j and ρ_j are not unique (as $\prod \delta_j = 1$) and may conveniently be normalized by setting $m_1 = \rho_1 = 0$. The principal series is obtained by defining (in analogy with the case

$n=2$) $(f, f) = \int |f(z)|^2 \prod_{j=2}^n dx_j dy_j$, where $z = \|x_j + iy_j\|$, and taking the ρ_j to be pure imaginary. More precisely, these are the "non-degenerate" representations in the principal series; there is also a set of "degenerate" representations which are limiting cases of the non-degenerate ones and which (relative to the present formulations) must be given in a slightly more complex form. These degenerate representations form a set of measure zero in the decomposition of the regular representation into the principal series given by the Plancherel formula, and do not occur in the case $n=2$ (in general the representations of the principal series depend on partitions of n as sums of positive integers, that partition in which the integers are all one corresponding to the non-degenerate representation). The complementary series has, as in the case $n=2$, the same expression for T_g but a more complex inner product between elements of the representation space than in the case of the principal series.

Instead of the space Z it is possible and for some purposes convenient to use a complex space \tilde{U} which is related to Z in somewhat the same way that the 2-sphere is related to the complex plane. \tilde{U} is defined as the quotient space of the subgroup U of unitary matrices in G by its center of diagonal unitary matrices in G . Here U is the maximal compact subgroup of G and its use facilitates the elaboration of the connection between a representation of G and the representation of U obtained by contraction of the given representation, e.g. the determination of the number of times this contraction contains a given irreducible representation of U . The use of U in connection with the unimodular group is based in the circumstance that every g in G has the form $g = ku$ with $u \in U$ and $k \in K$, u being unique within multiplication by an element of the center of U , so that the corresponding element \tilde{u} of \tilde{U} is actually unique. The non-degenerate representations of the principal series, e.g., can now be put in the form $(T_g f)(\tilde{u}) = f(\tilde{u}g) \psi(u, g)$, where the representation space is $L_2(U)$ (relative to Haar measure) and $\tilde{u}g$ is defined as \tilde{u}_1 where $ug = ku_1$. This representation space also is well adapted to the study of the "spherical" functions introduced by the authors, these being functions ω of the form $(T_g f, f)$, where f is a vector in the representation space invariant under the T_u with $u \in U$, so that $\omega(gu) = \omega(ug) = \omega(g)$ for $u \in U$.

Although a unitary operator on an infinite-dimensional Hilbert space never has a trace in the conventional sense, it is nevertheless possible to introduce a cogent character theory. If T is an irreducible unitary representation, then for any sufficiently smooth function f on G vanishing outside a compact set, $\int T_g f(g) dg$ is shown to be in the Hilbert-Schmidt class and has a finite trace. Moreover, there exists a function τ on G such that $\text{tr} \int T_g f(g) dg = \int \tau(g) f(g) dg$ for all such f ; this function τ displays the basic properties of the characters for finite-dimensional representations, and in particular two representations are unitarily equivalent if and only if they have identical characters. From this it follows that two representations in the principal series, depending on the parameters m_j, ρ_j and m'_j, ρ'_j are equivalent if and only if there is a permutation of the m_j into the m'_j which also takes the ρ_j into the ρ'_j .

For any locally compact group G , an integrable function f on G is determined by the operators $\int T_g f(g) dg$ as T ranges over the factor representations of G . In the case of the unimodular group the authors give the following formula, valid for sufficiently smooth functions f vanishing outside compact sets, expressing f in terms of such operators, where however

T need only range over the representations in the principal series: $f(e) = \int \text{tr } A_x dm(x)$, where e is the group identity, x denotes the parameters m_i and ρ_j on which the representation depends, $A_x = \int T_{e,x} f(g) dg$, and m is a certain measure absolutely continuous relative to the euclidean measure in the space of the $i\rho_1, \dots, i\rho_n$. From this analog to the Fourier integral theorem the following analog of the Plancherel

formula is readily deduced: $\int |f(g)|^2 dg = \int \text{tr } (A_x^* A_x) dm(x)$, f as before.
I. E. Segal (Chicago, Ill.).

Gelfand, I. M., and Naimark, M. A. Unitary representations of semisimple Lie groups. I. Amer. Math. Soc. Translation no. 64, 42 pp. (1952).
Translated from Mat. Sbornik N.S. 21(63), 405-434 (1947); these Rev. 9, 328.

NUMBER THEORY

✓Khinchin, A. Y. Three pearls of number theory. Graylock Press, Rochester, N. Y., 1952. 64 pp. \$2.00.
Translated by F. Bagemihl, H. Komm and W. Seidel from Tri žemčuziny teorii čisel, 2d ed., OGIZ, Moscow-Leningrad, 1948; cf. these Rev. 11, 83; 13, 13.

Thébault, Victor. Sur certaines puissances entières des nombres consécutifs. Mathesis 60, 248-252 (1951).

The author investigates the conditions on the base B of numeration under which the m th powers of the non-negative integers $< B$ terminate in B distinct digits in the unit's position. The results follow from simple congruence theorems.
D. H. Lehmer (Los Angeles, Calif.).

Roth, Klaus. Sur quelques ensembles d'entiers. C. R. Acad. Sci. Paris 234, 388-390 (1952).

Denote by $1 = u_1 < u_2 < \dots < u_k \leq x$ a sequence of integers no three of which form an arithmetic progression. Denote by $A(x)$ the maximum value of k . The author proves that $\lim_{x \rightarrow \infty} A(x)/x = 0$. This has been conjectured for about 20 years. Outline of the proof: Put

$$S = \sum_{j=1}^k e(\alpha u_j) \quad (e(x) = e^{2\pi i x})$$

where α is a real number. Determine integers h and g for which

$$\alpha = \frac{h}{g} + \beta, \quad g \leq x^{1/2}, \quad g|\beta| < x^{-1/2}.$$

Let $m < x$. Define

$$S' = \frac{A(m)}{mg} \left[\sum_{j=1}^g e\left(\frac{h}{g} \cdot \frac{j}{m}\right) \sum_{n=1}^m e(\beta n) \right].$$

The author proves that

$$(1) \quad |S - S'| < x m^{-1} A(m) - k + O(m x^{1/2}).$$

Put $\lim_{x \rightarrow \infty} A(x)/x = A$, and let m be so large that for $x > m$, $|A(x)/x - A| < \epsilon$. Let $1 \leq u_1 < u_2 < \dots < u_k \leq 2x$, $k = A(2x)$, be a maximal sequence not containing an arithmetic progression of three terms. $2v_1, 2v_2, \dots, 2v_l$ are the even numbers among the u 's. It is easy to see that $l \geq A(2x) - A(x)$. Thus

$$(2) \quad \frac{2x}{m} A(m) - k < 4\epsilon x, \quad \frac{x}{m} A(m) - l < 4\epsilon x.$$

Put

$$f(\alpha) = \sum_{j=1}^k e(\alpha u_j), \quad g(\alpha) = \sum_{j=1}^l e(\alpha v_j),$$

$$F(\alpha) = \frac{A(m)}{m} \sum_{n=1}^{2x} e(\alpha n), \quad G(\alpha) = \frac{A(m)}{m} \sum_{n=1}^x e(\alpha n).$$

Put $\eta = \epsilon^{-1/2} x^{-1}$. Using (1) and (2) the author proves that for $|\alpha| < \eta$

$$(3) \quad f(\alpha) - F(\alpha) = O(\epsilon x + m x^{1/2}), \quad g(\alpha) - G(\alpha) = O(\epsilon x + m x^{1/2}).$$

For $\eta < \alpha < 1 - \eta$ by (1) and (2)

$$(4) \quad f(\alpha) = O(\epsilon^{1/2} x + m x^{1/2}).$$

Now $u_i + u_j \neq 2u_r$. Thus

$$(5) \quad \int_{-\eta}^{1-\eta} f(\alpha) g^2(-\alpha) d\alpha = l \leq N.$$

Using (3) and (4) the author obtains by a simple calculation

$$(6) \quad \int_{-\eta}^{1-\eta} f(\alpha) g^2(-\alpha) d\alpha = (m^{-1} A(m))^2 N^2 + O(\epsilon^{1/2} N^2 + \epsilon^{-1/2} m N^{3/2}).$$

Thus from (5) and (6) $A^2 = O(\epsilon^{1/2})$ or $A = 0$, q.e.d.

P. Erdős (Los Angeles, Calif.).

de Bruijn, N. G. On the number of positive integers $\leq x$ and free of prime factors $> y$. Nederl. Acad. Wetensch. Proc. Ser. A. 54, 50-60 (1951).

Let $\Psi(x, y)$ denote the quantity specified in the title. It was proved by Dickman [Ark. Mat. Astr. Fys. 22, no. A10 (1930)] that if u is a fixed positive number, then $\lim_{y \rightarrow \infty} y^{-u} \Psi(y^u, y) = \rho(u)$, where $\rho(u)$ is a continuous function of u defined by the properties $\rho(u) = 1$ for $0 < u \leq 1$ and $u\rho'(u) = -\rho(u-1)$ for $u > 1$. (Dickman's result has been rediscovered recently by several other authors listed in the bibliography of the present paper). In another paper [J. Indian Math. Soc. (N.S.) 15, 25-32 (1951); these Rev. 13, 326] the author has made a detailed study of the function $\rho(u)$. In the present paper he constructs and studies a function $\Lambda(x, y)$ which is a closer approximation to $\Psi(x, y)$ than $x\rho(\log x / \log y)$. This function $\Lambda(x, y)$ is defined as $x \int_0^\infty \rho((\log x - \log t) / \log y) d([t]/t)$ if x is not an integer and as $\Lambda(x+0, y)$ if x is an integer. He proves that if $x > 0$ and $y \geq 2$ then $|\Psi(x, y) - \Lambda(x, y)| < Cx(\log x / \log y)^2 R(y)$, where $R(y)$ is any function such that

$$R(y) \downarrow 0 \quad (y \rightarrow \infty), \quad R(y) > y^{-1} \log y \quad (y \geq 2),$$

and $|\pi(y) - li y| < yR(y) / \log y \quad (y \geq 2)$. The method of proof is similar to that used by the author in an earlier paper [Nederl. Akad. Wetensch. Proc. 53, 803-812 = Indagationes Math. 12, 247-256 (1950); these Rev. 12, 11] to prove an analogous estimate for $\Phi(x, y)$, the number of positive integers $\leq x$ and free of prime factors $< y$. The author also applies his main result in proving that if $g(n)$ denotes the largest prime divisor of the positive integer n , then $\sum_{n=1}^x \log g(n) = ax \log x + O(x)$, where $a = \int_0^\infty (1+u)^{-2} \rho(u) du$.
P. T. Bateman (Urbana, Ill.).

Borel, Émile. Démonstration élémentaire du théorème de Dirichlet relatif aux nombres premiers d'une progression arithmétique. C. R. Acad. Sci. Paris 234, 769-770 (1952).

The author sketches a probabilistic argument leading to Dirichlet's theorem on primes in arithmetic progressions. The reviewer is unable to follow the method. P. Erdős.

Hammersley, J. M. The sums of products of the natural numbers. *Proc. London Math. Soc.* (3) 1, 435-452 (1951).

Let $\Pi_{r,s}$ denote the sum of the products of the first r natural numbers taken s at a time. Three problems are proposed: (i) To determine a formula from which $\Pi_{r,s}$ can be calculated, at least approximately, when r and s are given. (ii) Given r , to state what value or values of s maximize $\Pi_{r,s}$. (iii) To prove the conjecture that the value of s which maximizes $\Pi_{r,s}$ for given r is unique. The main result is that, for (ii),

$$s = r - \left[\rho + \frac{1}{2} + \frac{\zeta(2) - \zeta(3)}{\rho} + \frac{h}{\rho^2} \right],$$

where the square brackets denote the greatest integer function, $\rho = \log(r+1) + \gamma - \frac{1}{2}$, γ = Euler's constant, and $-1.1 < h < 1.5$. For (i), the author gives an exact expression for $\Pi_{r,s}$ as an "almost-triangular" determinant of order $r-s$, which is, however, useless for calculation. He suggests the approximate formula

$$\Pi_{r,s} \approx \frac{r!}{(r-s)!} e^{1-\gamma} \rho^{r-s-1} \left\{ \rho + \frac{1}{2}(r-s) \right\},$$

(ρ defined as above), if s is about equal to $r - \log r$. In a note added in proof, he states that Erdős has proved (iii) for all sufficiently large r . *N. J. Fine* (Philadelphia, Pa.).

Apostol, T. M. Addendum to 'On the Lerch zeta function.' *Pacific J. Math.* 2, 10 (1952).

This note refers to a paper in *Pacific J. Math.* 1, 161-167 (1951) [these Rev. 13, 328]. Additional references are given, and a misprint is corrected. *N. G. de Bruijn* (Delft).

Apostol, T. M. Theorems on generalized Dedekind sums. *Pacific J. Math.* 2, 1-9 (1952).

The author gives a second proof for his reciprocity law for generalized Dedekind sums of the form

$$s_p(h, k) = \sum_{\mu=1}^{k-1} \mu k^{-1} B_p(\{h\mu/k\}),$$

where $(h, k) = 1$; B_p denotes the p th Bernoulli polynomial, and $\{x\} = x - [x]$. [See Apostol, *Duke Math. J.* 17, 147-157 (1950); these Rev. 11, 641.] He expresses $s_p(h, k)$ in terms of the Hurwitz zeta function $\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$ by

$$s_p(h, k) = ip!(2\pi i k)^{-p} \sum_{\mu=1}^{k-1} \cot(\pi h\mu/k) \zeta(p, \mu/k)$$

(p odd, $p > 1$). Application of the theory of residues to the function $f(z) = \cot(\pi z) \cot(\pi hz/k) \zeta(p, z/k)$ finishes the proof. *N. G. de Bruijn* (Delft).

Bailey, W. N. A note on two of Ramanujan's formulae. *Quart. J. Math., Oxford Ser.* (2) 3, 29-31 (1952).

In an unpublished manuscript, Ramanujan gave an alternate proof of his famous formula

$$(1) \quad \sum_{n=0}^{\infty} p(5n+4)x^n = 5 \prod_{n=1}^{\infty} \frac{(1-x^{2n})^5}{(1-x^n)^6},$$

where $p(n)$ is the number of partitions of n . This proof is based on the (unproved) formula

$$(2) \quad x \prod_{n=1}^{\infty} \frac{(1-x^{2n})^5}{(1-x^n)^6} = \sum_{n=0}^{\infty} \left[\frac{x^{2n+1}}{(1-x^{2n+1})^2} - \frac{x^{2n+2}}{(1-x^{2n+2})^2} - \frac{x^{2n+3}}{(1-x^{2n+3})^2} + \frac{x^{2n+4}}{(1-x^{2n+4})^2} \right].$$

The author supplies a proof of (2), using the known sum of a well-poised basic bilateral hypergeometric series. From this same sum he also deduces another result of Ramanujan's:

$$(3) \quad \prod_{n=1}^{\infty} \frac{(1-x^n)^5}{(1-x^{2n})^6} = 1 - 5 \sum_{n=0}^{\infty} \left[\frac{(5m+1)x^{2m+1}}{1-x^{2m+1}} - \frac{(5m+2)x^{2m+2}}{1-x^{2m+2}} - \frac{(5m+3)x^{2m+3}}{1-x^{2m+3}} + \frac{(5m+4)x^{2m+4}}{1-x^{2m+4}} \right],$$

proofs of which have been given by Darling and by Mordell. *N. J. Fine* (Philadelphia, Pa.).

Giuga, Giuseppe. Su una presumibile proprietà caratteristica dei numeri primi. *Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat.* (3) 14(83), 511-528 (1950).

Let $S_r(n) = 1^r + 2^r + \dots + (n-1)^r$. The author observes that, by Fermat's theorem, $(1) S_{n-1}(n) \equiv -1 \pmod{n}$ in case n is a prime and asks whether (1) is characteristic of primes so that (1) is false for all composite numbers n . He proves that any composite solution of (1) must have more than 1000 decimal digits and concludes that it is extremely probable that (1) holds only for primes. *D. H. Lehmer*.

Tatuzawa, Tikao, and Iseki, Kanesiroo. On Selberg's elementary proof of the prime-number theorem. *Proc. Japan Acad.* 27, 340-342 (1951).

Let $\Lambda(n) = \log p$ for $n = p^1$ (p prime) and 0 otherwise. Put $\psi(x) = \sum_{n \leq x} \Lambda(n)$. Applying the Möbius inversion formula, the author establishes the formula

$$\psi(x) \log x + \sum_{n \leq x} \psi(x/n) \Lambda(n) = 2x \log x + O(x).$$

The analysis resembles Shapiro's derivation [*Ann. of Math.* (2) 51, 485-497 (1950); these Rev. 11, 419] of Selberg's lemma [*Ann. of Math.* (2) 50, 305-313 (1949); these Rev. 10, 595]. The author's result may be used in place of Selberg's lemma to prove the prime number theorem.

A. L. Whiteman (Los Angeles, Calif.).

Fogels, E. K. Analogue of the Brun-Titchmarsh theorem. *Latvijas PSR Zinātņu Akad. Fiz. Mat. Inst. Raksti.* 2, 46-58 (1950). (Russian. Latvian summary)

The theorem mentioned in the title is the result that the number of primes not exceeding x in an arithmetic progression with difference $k \leq x^a$, $0 < a < 1$, is $O(x / \{\varphi(k) \log x\})$. Here, the author proves a result formulated as follows. Let $f = ax^2 + buv + cv^2$ be a positive definite form with discriminant $-D = b^2 - 4ac$, let $h(-D)$ be the number of classes of forms with discriminant $-D$, and let $\pi(x, f)$ be the number of primes not exceeding x which are represented by the form f . Then, corresponding to each $\epsilon > 0$ there is a number $D_0(\epsilon)$ such that if $D > D_0(\epsilon)$ and if $D < x^{1/\epsilon}$ then $\pi(x, f) < \alpha x / \{h(-D) \log x\}$ where

$$\alpha = (1 + \epsilon) \cdot (5 - 2.1 \log D / \log x - \log \log x / \log x)^{-1}.$$

The principal tool in the proof is a ready extension of A. Selberg's sieve method from the case of rational integers to the case of ideals in $K(\sqrt{-D})$. *L. Schoenfeld*.

Carlitz, L. Independence of arithmetic functions. *Duke Math. J.* 19, 65-70 (1952).

The author establishes the algebraic independence of various arithmetical functions in a sense different from that considered by Bellman and Shapiro [same J. 15, 229-235 (1948); these Rev. 9, 500] and by Wade [ibid. 15, 237 (1948); these Rev. 9, 500]. The present sense is based on

the Dirichlet multiplication of two arithmetical functions. If $I_k(m) = m^k$, then various simple arithmetical functions are representable symbolically in terms of I_0 and I_1 ; thus $\mu = (I_0)^{-1}$, $d = I_0^2$, $\sigma = I_0 I_1$, $\phi = I_1 (I_0)^{-1}$. The author first proves that there is no symbolic identity of the form $\Phi(I_0, I_1, \dots, I_r) = 0$, where Φ is a polynomial, and then establishes similar results involving power series. The proofs are elementary and of a purely arithmetical character.

H. Davenport (London).

Cole, A. J. On the product of n linear forms. *Quart. J. Math., Oxford Ser. (2)* 3, 56-62 (1952).

Let L_1, L_2, \dots, L_n be n homogeneous linear forms in n variables u_1, u_2, \dots, u_n with real coefficients and determinant $\Delta \neq 0$ and let c_1, c_2, \dots, c_n be any n real numbers. Then the author proves that there exist integers u_1, u_2, \dots, u_n , such that

$$(1) \quad L_1 + c_1 > 0, \quad L_2 + c_2 > 0, \quad \dots, \quad L_{n-1} + c_{n-1} > 0,$$

$$(2) \quad (L_1 + c_1)(L_2 + c_2) \cdots (L_{n-1} + c_{n-1}) | L_n + c_n | \leq \frac{1}{2} |\Delta|.$$

He proves that this result is best possible and he deduces sufficient conditions for L_1, \dots, L_n such that (1) and (2) have an infinity of integer solutions (u_1, \dots, u_n) . These theorems are analogous to former theorems of Chalk [same *J.* 18, 215-227 (1947); 19, 67-80 (1948); these *Rev.* 9, 413; 10, 18], who considered the inequalities $L_1 + c_1 > 0, \dots, L_n + c_n > 0$ instead of (1), in which case the number $\frac{1}{2} |\Delta|$ in (2) must be replaced by $|\Delta|$.

J. F. Koksma.

Brandt, H. Das quadratische Reziprozitätsgesetz im Gausschen Zahlkörper. *Comment. Math. Helv.* 26, 42-54 (1952).

It is shown that the quadratic reciprocity law for the Gaussian number field can be formulated as follows: Let δ be an odd Gaussian prime, that is the discriminant of a binary quadratic form, and let π be any Gaussian prime. Then $[\delta/\pi] = [\pi/\delta]$, where the right side is the character of π modulo δ , and the left side is 1 if π can be represented by a quadratic form of discriminant δ , -1 otherwise.

W. H. Mills (New Haven, Conn.).

Kneser, Martin. Zum expliziten Reziprozitätsgesetz von I. R. Šafarevič. *Math. Nachr.* 6, 89-96 (1951).

In his presentation of I. R. Šafarevič's theory of the explicit form for the law of reciprocity [Mat. Sbornik N.S. 26(68), 113-146 (1950); these *Rev.* 11, 230] H. Hasse [Math. Nachr. 5, 301-327 (1951); these *Rev.* 13, 113] announced a direct verification of Šafarevič's formulae by the author as a consequence of the general theorems for the norm residue symbol applied to p -adic fields K_p . These detailed computations are carried out in this paper. The customary reductions lead to emphasis of the special case in which K_p contains all p^n -th roots of unity, $p|p$. First the classical norm residue symbol (A, B) is evaluated for a fixed prime element $A = \pi$ of K_p and a unit $B = \epsilon$ of K_p . Using Hasse's basis representation [see Hasse's version, loc. cit.] for the units ϵ as $\prod_i E(\alpha_i, \pi^i) E^*(\alpha)$, modulo p^n -th powers in K_p , $1 \leq i \leq (pe)/(p-1)$, e the ramification degree of K_p over the rational p -adic field, $(i, p) = 1$, where the terms $E(\dots)$, $E^*(\dots)$ are special units (with arguments in the valuation ring of the maximal absolutely unramified subfield of K_p) given by Šafarevič as infinite products, the author shows that $(\pi, \epsilon) = (\pi, E^*(\alpha)) = \rho_{\pi}^{\text{Sp}(\alpha)}$, Sp denoting the trace taken from the maximal unramified subfield to the rational p -adic field. It is to be noted that $(\pi, E(\alpha_i, \pi^i)) = 1$ as a

consequence of the formula $(A, 1-A) = 1$ ($A \neq 0, 1$) for the norm residue symbol, which does not play a substantial role in the general multiplicative theory of the latter. Next the generalization is carried out for arbitrary non-zero $A, B \in K_p$ by means of an interesting evaluation of $(1-x, 1-y)$ for $x, y \in p$. For a different formula for (A, B) see also W. H. Mills [Amer. J. Math. 73, 65-77 (1951); these *Rev.* 12, 592] who does not use the full theory of units employed in this paper.

O. F. G. Schilling (Chicago, Ill.).

Lang, Serge. On quasi algebraic closure. *Ann. of Math.* (2) 55, 373-390 (1952).

A field F is called C_i if every form in F in n variables and degree d with $n > d^i$ has a non-trivial zero in F ; C_0 is Artin's quasi-algebraic closedness; for general C_i cf. C. Tsen, J. Chinese Math. Soc. 1, 81-92 (1936). If F is C_i and admits a normic form of order i (= a form with $n = d^i$ possessing no non-trivial zero in F) then an extension of transcendence h over F is C_{i+h} ; this refines Tsen [loc. cit.] and the proof uses Tsen's argument as well as a lemma of Artin. Modifications for (single or several) polynomials without constant term (as in Tsen, loc. cit.) are given. The power series field $k\{t\}$ over a finite field k is C_2 ; the proof depends on Chevalley's theorem [Abh. Math. Sem. Univ. Hamburg 11, 73-75 (1935)]. A complete valued field with algebraically closed residue field, the maximal unramified extension of a complete field with perfect residue field (Artin's conjecture), the absolutely algebraic subfield of the maximal unramified extension of an ordinary p -adic field, and the field of convergent power series over an algebraically closed valued constant field are all C_1 ; the proof uses expansion structures of the fields and some approximation and specialization processes for forms and zeros. Application to the cohomology approach to local class field theory [Hochschild, same *Ann.* 51, 331-347 (1950); these *Rev.* 11, 490] is observed; the treatment avoids the direct proof of the 2nd inequality but proceeds similarly as in the hypercomplex approach.

T. Nakayama (Nagoya).

Morishima, Taro. On Fermat's last theorem (thirteenth paper). *Trans. Amer. Math. Soc.* 72, 67-81 (1952).

In this paper the author continues his own and Vandiver's investigations of the first case of Fermat's last theorem, that is, he considers the equation (*) $\alpha^l + \beta^l + \gamma^l = 0$ where α, β, γ are integers in the cyclotomic field $k(\zeta)$ that are prime to $1-\zeta$, l is an odd prime and ζ is a primitive l th root of unity. Thus he proves that, under the conditions just mentioned, at least one of the first half of the set of Bernoulli numbers $B_1, B_2, \dots, B_{(l-3)/2}$ must be divisible by l if (*) has a solution; he also proves that at least seven of this set of Bernoulli numbers must be divisible by l . As another result he proves that if α, β, γ are integers in the field $k(\zeta + \zeta^{-1})$ prime to $1-\zeta$, then the first factor of the class number of $k(\zeta)$ is divisible by l^3 ; he had previously proved that it is divisible by l^3 [Jap. J. Math. 11, 241-252 (1935), p. 251]. There are also results concerning the irregular class group; thus it is proved that if in a normal basis of this group the number of quadratic non-residues (mod l) exceeds the number of quadratic residues by less than seven, then Fermat's equation (*) cannot be satisfied under the conditions given above.

H. W. Brinkmann (Swarthmore, Pa.).

Tannaka, Tadao. Some remarks concerning p -adic number field. *J. Math. Soc. Japan* 3, 252-257 (1951).

Let K be of finite degree over the p -adic rationals (arguments used don't generalize beyond this case), let k and k'

be subfields of K with K/k and K/k' normal, let \mathcal{G} and \mathcal{G}' be the Galois groups, and let G and G' be the groups of elements $A \in K$ and $A' \in K'$ with $N_{K/k}A = 1$ and with $N_{K/k'}A' = 1$, and let (a) be a factor set on \mathcal{G} to K which corresponds to a division algebra $(K/k, a(\sigma, \tau))$. Theorems: (1) If K/k and K/k' are abelian, then $k' \subset k \subset G \subset G'$, with " \subset " denoting proper inclusion. (2) If K/k is abelian then G is generated by the elements $s(\sigma, \tau)/a(\tau, \sigma)$ and $b^{1-\sigma}$, for $\sigma, \tau \in \mathcal{G}$ and $b \in K$. (2') For arbitrary \mathcal{G} , the group G is generated by the elements $b^{1-\sigma}$ and the elements $a'(\sigma, \tau)/a'(\tau, \sigma)$ for factor sets (a') similar to (a) which satisfy the condition $\prod_{\tau \in \mathcal{G}} a'(\sigma\tau, \gamma) = \prod_{\tau \in \mathcal{G}} a'(\tau\sigma, \gamma)$. (3) If K/k is abelian and G is generated by the elements $b^{1-\sigma}$, then K/k is cyclic. (4) If the commutator factors \mathcal{G}/\mathcal{G}' , $\mathcal{G}'/\mathcal{G}''$, ... are all cyclic, then G is generated by the elements $b^{1-\sigma}$.

This paper proves (1), (2), and (2') and reports that (1) and (2) were first proved by T. Tannaka [Shijō-Sōgaku-Danwakai 236, 1050-1056 (1942)], (2') by T. Nakayama [ibid. 247, 1596-1602 (1942)], (3) by Y. Matsushima [ibid. 252, 213-217 (1943)], and (4) by H. Kuniyoshi [ibid. (2) 15, 534-536, 536-538 (1949)], all these papers being in Japanese. Related results: H. Kuniyoshi, [Tōhoku Math. J. (2) 1, 186-193 (1950); these Rev. 11, 711], Nakayama and Matsushima [Proc. Imp. Acad. Tokyo 19, 622-628 (1943); these Rev. 7, 238], and S. Wang [Amer. J. Math. 72, 323-334 (1950); these Rev. 11, 577]. Misprints: p. 254 line 8, read " ϵ ", " $\bar{\epsilon}$ " for " e ", " \bar{e} "; line 14, read "inclusion" for "implication"; line 19, read " $N_{K/k}$ " for " N ".

G. Whaples (Bloomington, Ind.).

Inaba, Eizi. Number of divisor classes in algebraic function fields. Proc. Japan Acad. 26, no. 7, 1-4 (1950).

The author proves an analogue to the theorem of Siegel and Heilbronn on the class number of imaginary quadratic fields. Let k be a (fixed) field of q elements which serves as the coefficient field of the algebraic function fields K of one variable under consideration. Set $A = (\log h)/(g \log q)$ where h denotes the number of divisor classes of degree 0 and g the

genus of the function field. Then $\lim_{p \rightarrow \infty} A = 1$ provided only such fields K are considered which are of bounded fixed degree over some rational subfield. The proof of $\liminf A \geq 1$ does not require the latter restriction; it is made to depend on a modification of H. Reichardt's estimates for the number of prime divisors of a given degree [Math. Z. 40, 713-719 (1936)] by means of the Riemann hypothesis for the fields K . For $1 \geq \limsup A$ estimates for the terms of the product expansion for the ζ -function of K are carried out assuming the restriction on the class of fields K .

O. F. G. Schilling (Chicago, Ill.).

Gel'fond, A. O. The approximation of algebraic numbers by algebraic numbers and the theory of transcendental numbers. Amer. Math. Soc. Translation no. 65, 45 pp. (1952).

Translated from Uspehi Matem. Nauk (N.S.) 4, no. 4(32), 19-49 (1949); these Rev. 11, 231.

Gel'fond, A. O. On the algebraic independence of transcendental numbers of certain classes. Amer. Math. Soc. Translation no. 66, 46 pp. (1952).

Translated from Uspehi Matem. Nauk (N.S.) 4, no. 5(33), 14-48 (1949); these Rev. 11, 231.

Obrechhoff, N. Sur l'approximation diophantique linéaire. C. R. Acad. Bulgare Sci. 3, no. 2-3 (1950), 1-4 (1951). (French. Russian summary)

Let $\omega_1, \dots, \omega_m$ be real, and let n be a positive integer. It is well known that there exist integers x_1, \dots, x_m (not all 0) and y such that $|x_i| \leq n$ and

$$|\omega_1 x_1 + \dots + \omega_m x_m - y| \leq (n+1)^{-n}.$$

The author proves that the sign of equality is necessary in the last inequality if and only if $\omega_1, \dots, \omega_m$ are a permutation of $\lambda_1/(n+1), \lambda_2/(n+1)^2, \dots, \lambda_m/(n+1)^m$, where $\lambda_1, \dots, \lambda_m$ are integers relatively prime to $n+1$.

H. Davenport (London).

ANALYSIS

Hardy, G. H., Littlewood, J. E., and Pólya, G. Inequalities. 2d ed. Cambridge, at the University Press, 1952. xii+324 pp. \$4.75.

A reprint with minor changes of the first edition [Cambridge, 1934]. Three appendices have been added.

***Orloff, Constantin.** Les spectres des nombres qui ne sont pas entiers. Premier Congrès des Mathématiciens et Physiciens de la R.P.F.Y., 1949. Vol. II, Communications et Exposés Scientifiques, pp. 73-81. Naučna Knjiga, Belgrade, 1951. (Serbo-Croatian. French summary)

***Orloff, Constantin.** Les spectres mathématiques. Premier Congrès des Mathématiciens et Physiciens de la R.P.F.Y., 1949. Vol. II, Communications et Exposés Scientifiques, pp. 83-97. Naučna Knjiga, Belgrade, 1951. (Serbo-Croatian. French summary)

These two expository papers review and extend the so-called spectral theory of M. Petrovich in which a homomorphism is established between power series and the decimal representation of real numbers [see Jahrbuch über die Fortschritte der Mathematik 47, 320-321 (1924)]. The author considers the extension of the idea of Petrovich to algebra, analysis, number theory and probability.

D. H. Lehmer (Los Angeles, Calif.).

Ryll-Nardzewski, C. Certains théorèmes des moments. Studia Math. 12, 225-226 (1951).

Si Γ est une image rectifiable homéomorphe du segment $(0, 1)$ dans le plan complexe, n'ayant pas plus d'un point dans le cercle $|z| \leq \rho$, et si $|\int_{\Gamma} e^{f(z)} dz| = O(\rho^n)$ (f sommable sur Γ), f est nulle sur Γ presque partout. L. Schwartz.

Každan, Ya. M. On the moment problem for J_p -matrices. Doklady Akad. Nauk SSSR (N.S.) 82, 329-332 (1952). (Russian)

The author proves two theorems. The first is equivalent to a theorem of Kreĭn [same Doklady (N.S.) 69, 125-128 (1949); these Rev. 11, 670], but is proved in an elementary way. Let A be a regular J_p -matrix, which is a generalization of a Jacobi matrix: its elements are p -rowed square matrices A_{ik} , with $A_{ii} = 0$ for $|i-k| > 1$, $A_{i,i+1}$ nonsingular. A matrix function $T(\lambda)$, with elements $t_{ij}(\lambda)$, whose values are p -rowed square matrices, is called increasing if for every interval $(\lambda, \lambda + \Delta)$ the quadratic form with coefficients $t_{i+j}(\lambda + \Delta) - t_{ij}(\lambda)$ is nonnegative. Theorem: Let $D_b(\lambda)$ be the matrix polynomials satisfying $D_{-1}(\lambda) = 0$, $D_0(\lambda) = J$, the unit matrix, and

$$A_{k,k-1}D_{k-1}(\lambda) + A_{k,k}D_k(\lambda) + A_{k,k+1}D_{k+1}(\lambda) = \lambda D_k(\lambda).$$

Then there is at least one symmetric increasing $T(\lambda)$ such that

$$\int_{-\infty}^{\infty} D_i(\lambda) D_j(\lambda) dT(\lambda) = \delta_{ij} J.$$

The second theorem deals with an ordinary Jacobi matrix with positive nondiagonal elements. Let $\sigma(\lambda)$ be its spectral function, $\sigma(-\infty) = 0$. If $a_{h,h+1} < M$ and $a_{h,h} \rightarrow \infty$, the spectral function is unique, the spectrum is discrete, and is bounded on the left.

R. P. Boas, Jr. (Evanston, Ill.).

Novodvorskiĭ, E. P., and Pinsker, I. Š. The process of equating maxima. *Uspehi Matem. Nauk (N.S.)* 6, no. 6(46), 174-181 (1951). (Russian)

The authors establish the validity of a process, attributed to Ya. L. Remez, for constructing the function of a given class which deviates least from a given function. The class is required to have the following properties: it consists of functions $\Delta(t)$ continuous on a closed interval, such that (a) two functions are identical if they coincide at more than n points (a double zero of the difference counting twice), (b) a function of the class is determined when its values at $n+1$ points are assigned and depends continuously on the points.

R. P. Boas, Jr. (Evanston, Ill.).

Morozov, M. I. On certain questions of the uniform approximation of continuous functions by means of functions from interpolation classes. *Izvestiya Akad. Nauk SSSR. Ser. Mat.* 16, 75-100 (1952). (Russian)

This paper contains detailed proofs of theorems announced earlier [*Doklady Akad. Nauk SSSR (N.S.)* 77, 381-383 (1951); these Rev. 12, 680], together with preliminary material on general properties of functions belonging to interpolation classes [definition in the cited review].

R. P. Boas, Jr. (Evanston, Ill.).

Theory of Sets, Theory of Functions of Real Variables

Bachmann, H. Vergleich und Kombination zweier Methoden von Veblen und Finsler zur Lösung des Problems der ausgezeichneten Folgen von Ordnungszahlen. *Comment. Math. Helv.* 26, 55-67 (1952).

The problem of distinguished sequences has been solved for a certain segment of the second number class by Veblen [*Trans. Amer. Math. Soc.* 9, 280-292 (1908)] with the aid of a transfinite sequence of normal functions, for a larger segment of that class by Bachmann [*Vierteljahr. Naturforsch. Ges. Zürich* 95, 115-147 (1950); these Rev. 12, 165] with a generalization of Veblen's method, and for another segment of that class by Finsler [*Comment. Math. Helv.* 25, 75-90 (1951); these Rev. 13, 120] with the aid of a transfinite sequence of arithmetical operations. The author shows that Finsler's segment is much smaller than Veblen's, that Veblen's (Bachmann's) method can be combined with Finsler's, but that this combination solves the problem of distinguished sequences for the same segment of the second number class as Veblen's (Bachmann's) method alone.

F. Bagemihl (Rochester, N. Y.).

Andreoli, Giulio. Osservazioni sugli insiemi perfetti lineari ed i sistemi di numerazione. *Ricerca, Napoli* 2, no. 2, 11-17 (1951).

If one considers the set of all dyadic ω -sequences ordered lexicographically one obtains a set S isomorphic with a perfect

linear set $S_0 \subseteq [0, 1]$. By identifying both points in each or in some sets of consecutive points in S one gets all possible types of corresponding perfect sets $S_0 \subseteq [0, 1]$.

Đ. Kurepa (Zagreb).

Harrington, W. J. A note on the denumerability of the rational numbers. *Amer. Math. Monthly* 58, 693-696 (1951).

In this paper, the author sets up a one-to-one correspondence between the rational numbers and the positive integers which makes possible a simpler formulation of the algorithm of Johnston [same Monthly 55, 65-70 (1948); these Rev. 9, 416]. By means of two transformations $S(a/b)$ and $T(a/b)$, defined as $a/(a+b)$ and $(a+b)/b$ respectively, where a/b is in lowest terms, the positive rationals are exhibited in an array where the first half of each row beyond the first results from the application of S to the successive elements of the preceding row, and the second half from the corresponding application of T . The counting system constructs a one-to-one correspondence between the non-negative rational numbers a/b and the positive integers N so that, given a/b , we can find N , and given N , we can find a/b . The correspondence can be extended to include the negative rationals in the same manner as given by Johnston.

W. H. Gage.

Monna, A. F. Sur une transformation simple des nombres P -adiques en nombres réels. *Nederl. Akad. Wetensch. Proc. Ser. A.* 55 = *Indagationes Math.* 14, 1-9 (1952).

Let P be a prime, R_+^* the set of all positive numbers, and $K^*(P)$ the set of all P -adic numbers not of the form P^{-N} where N and ν are rational integers. Every $a \in R_+^*$ has a unique representation $a = \sum_{n=0}^{\infty} a_n P^{-n-1}$ ($a_n =$ one of the digits $0, 1, \dots, P-1$ for all n) where infinitely many $a_n \neq 0$. The transformation $T: a \rightarrow \alpha = \sum_{n=0}^{\infty} a_n P^n$ maps R_+^* one-to-one on $K^*(P)$. The inverse T^{-1} is continuous on $K^*(P)$, but T itself is discontinuous at those rational a the denominator of which is a power of P , is continuous to the left at these points, and continuous everywhere else. The author discusses generalizations of T , and uses it to develop a measure theory on the set of all P -adic numbers.

K. Mahler (Manchester).

Aumann, Georg. Sind die elementargeometrischen Figuren Mengen? *Elemente der Math.* 7, 25-28 (1952).

Hellmich, Kurt. Stetige und halbstetige Punkt-Mengen-Funktionen. *Monatsh. Math.* 55, 265-296 (1951).

Let $F(a)$ be a function which associates with each point a of a set E a subset $F(a)$ of a set R . Kuratowski [*Fund. Math.* 18, 148-159 (1932)] has introduced the concept of such a function being upper semi-continuous or lower semi-continuous when the sets $F(a)$ are closed and non-empty. The present author generalizes the definitions of Kuratowski so as to obtain definitions of what he calls upper and lower semi-continuity in the strong sense and in the weak sense. He then obtains thirty-five theorems using these new definitions. These theorems, and their applications to functions of a real variable, have considerable interest, but the complexity of the definitions precludes a detailed discussion in a review.

D. W. Hall (College Park, Md.).

Haupt, Otto, et Pauc, Christian. La topologie approximative de Denjoy envisagée comme vraie topologie. *C. R. Acad. Sci. Paris* 234, 390-392 (1952).

On a set R a complete measure μ (finite or denumerably infinite) and a basis \mathfrak{B} for differentiation in the sense of de Possel are to be defined such that the domain of \mathfrak{B} (i.e. the

set of points of R admitting contracting sequences) equals R . Let $X \subset R$; a point $x \in X$ is called by the authors "approximately interior to X " or " D -interior to X " if the interior \mathfrak{B} -density of X exists at x and equals 1. The set of D -interior points of X is designated by $I(X)$. The approximately semi-continuous (or continuous) functions are identical with the D -semicontinuous (or D -continuous) functions. If f is a μ -measurable function bounded on the constituents of \mathfrak{B} and D -continuous at x , then the Lebesgue integral of f has a \mathfrak{B} -derivative at x which equals $f(x)$.

The so called weak Vitali property is, for \mathfrak{B} , equivalent to the property that, for every X , $I(X)$ is a μ -measure-kernel of X . This property is now always assumed. Then the sets of zero μ -measure, the nowhere dense sets, and the sets of first category coincide. If \mathfrak{N} designates the family of these sets, then every μ -measurable set M equals $I(M)$ (mod \mathfrak{N}). The D -closure of X is a measure-cover of X , and hence the measurable sets are identical with the D -measurable sets in the Jordan sense. Every μ -measurable function f becomes D -continuous if one takes a suitable set of \mathfrak{N} away from the domain of f . The Lebesgue integral is now equivalent to an integral of the Riemann type defined by means of a denumerable Jordan D -partition [see Pauc, same C. R. 230, 810-811 (1950); these Rev. 11, 587]. For the particular case of the Euclidean spaces it is stated that the family of the D -open sets is richer than the family of the open sets and that the D -topology is finer than the Euclidean topology.

A. Rosenthal (Lafayette, Ind.).

Enomoto, Shizu. On the notion of measurability. Proc. Japan Acad. 27, 208-213 (1951).

The author gives a new definition of measurability with regard to a Carathéodory outer measure μ in a metric space X . It is always assumed that $\mu(A) < +\infty$ for every bounded set A ; μ is then called a "conditionally finite outer measure". A completely additive class \mathfrak{M} on which μ is completely additive is called by the author a " μ -completely additive class". Let $\{K_n\}$ be a monotone increasing sequence of bounded sets measurable in the Carathéodory sense with $\bigcup K_n = X$ and let $\mathfrak{R}(\mu)$ be the class of sets A for which $\mu(A \cdot K_n) = \mu(K_n) - \mu(K_n - A)$. Let m be any (ordinary) regular outer measure such that $m(K_n) = \mu(K_n)$ and $m(A) \geq \mu(A)$ for every $A \in X$. Then m is called by the author a "dominant measure of μ " and any set $A \in \mathfrak{R}(m)$ is said to be an " m -dominant measurable set of μ ". Set $m_0(A) = \inf m_n(A)$ where the infimum is taken for all dominant measures m_n of μ . If μ_0 is itself an (ordinary) regular outer measure, then μ is called "relatively regular" (which in general does not imply the regularity of μ) and any μ_0 -dominant measurable set of μ is called a measurable set. The author proves that $\mathfrak{R}(m)$ is a μ -completely additive class, and moreover: In order that μ be relatively regular, it is necessary and sufficient that $\mathfrak{R}(\mu)$ be a μ -completely additive class.

A. Rosenthal (Lafayette, Ind.).

Jurkat, Wolfgang. Zur Bewegungsinvarianz des Lebesgueschen Masses. Math. Z. 54, 343-346 (1951).

In the n -dimensional Euclidean space, let $|W| = a^n$ be the elementary content of the cube W whose edge has the length a . Let $\mathfrak{U}(M)$ be a covering of the set M by countably many open cubes W_i (not necessarily parallel to the axes) and set its measure $m(\mathfrak{U}(M)) = \sum_i |W_i|$. Then, as usually, the outer measure of M is defined by $m(M) = \inf m(\mathfrak{U}(M))$, hence it is invariant with respect to motions. The author now shows in a simple manner that for every (closed) cube W one has $m(W) = |W|$.

A. Rosenthal (Lafayette, Ind.).

Stein, S. A measure-theoretic relation between a function and its reciprocal. Amer. Math. Monthly 58, 691-693 (1951).

Let X and Y be two bounded subsets of the Euclidean n -space. Let 2^Y denote the set of subsets of Y and let f be a function on X into 2^Y . The author considers the inverse function $f^{-1}(y)$ on Y into 2^X (which he calls the "reciprocal" function), designates the Lebesgue measure by m' , the Lebesgue integral by \int' , while he uses m for the Jordan content and \int for the Riemann integral, and proves the theorem: If $f(x)$ is Jordan measurable (he uses the expression "Riemann measurable") for each $x \in X$ and $f^{-1}(y)$ is Lebesgue measurable for each $y \in Y$, then the Lebesgue integral $\int_X m'f(x)dm'$ and the Riemann integral $\int_Y m f^{-1}(y)dm$ exist and are equal. Finally he gives some applications of this theorem.

A. Rosenthal (Lafayette, Ind.).

Aquaro, Giovanni. Sopra le formule di cambiamento di variabili negli integrali secondo Riemann. Rend. Sem. Fac. Sci. Univ. Cagliari 20 (1950), 193-206 (1951).

Generalizing the formula for the change of variables in multiple Riemann integrals the author obtains analogous formulas for the transformation of the upper and lower integrals of a bounded function f on a bounded set A provided that the transformation $y = \varphi(x)$ in the r -dimensional Euclidean space is "regular", i.e., satisfies certain conditions stated by the author. A generalization of the Jacobian plays an essential rôle here; i.e., there has to exist a bounded, non-negative, integrable function $J(y)$ such that for each set $U \subset A$ the content $c(U) = \int_{\varphi(U)} J(y)dy$. The author also shows that the usual conditions for the transformation of a multiple Riemann integral establish such a "regular" correspondence.

A. Rosenthal (Lafayette, Ind.).

*Picone, Mauro. Teoria moderna dell'integrazione delle funzioni. Lezioni d'analisi tenute nell'anno accademico 1945-46 raccolte dal dott. F. Mammana. Scuola Normale Superiore di Pisa. Quaderni matematici, no. 1. Libreria Goliardica, Pisa, undated. 271 pp. 600 Lire.

This book contains a detailed discussion of the r -dimensional Stieltjes integrals. After a short introduction concerning physical applications, chapter I defines the total, positive, and negative variations of an interval function $\alpha(T)$ in the r -dimensional space S , and the functions $\alpha(T)$ of finite variation (i.e. the total variation $V_\alpha(T)$ is to be finite). An interval function $\alpha(T)$ which is additive and of finite variation is called a "determining function (funzione determinante)" by the author. Then in chapter II the Riemann-Stieltjes integral of a point function $f(p)$ with respect to the determining function $\alpha(T)$ on a bounded set I is studied. In chapter III the following generalization is introduced. If any set I and any function $f(p)$ are given, designate by $[C]_I$ the class of the closed, bounded subsets of I on which $f(p)$ is continuous. For $C \in [C]_I$ consider $\int_C f(p)d\alpha$, introduce an order according to $C_1 \subset C_2$; if, regarding this order, there exists a limit of that integral for $C \rightarrow I$, then it is called by the author the "absolute integral" of $f(p)$ on I with respect to $\alpha(T)$. Moreover, if the "absolute integral" $\int_I |f(p)|dV_\alpha$ is finite, $f(p)$ is called by him "summable" on I with respect to $\alpha(T)$. In particular, he defines the "measure" of I with respect to $\alpha(T)$, $m_\alpha I$, by the integral $\int_I dV_\alpha$. For a bounded, open set A one has $m_\alpha(A) = \lim \sum_{i=1}^n V_\alpha(T_i)$ where the T_i belonging to a subdivision of the space are contained in A and the limit refers to a refinement of the subdivisions. A set I is called by the author a "Lebesgue set" if to every

$\epsilon > 0$ one can find a closed set $C_\epsilon \subset I$ and an open set $A_\epsilon \supset I$ such that $m_\alpha(A_\epsilon - C_\epsilon) < \epsilon$. In chapter IV the theory of this measure m_α is developed analogously to the classical theory of the Lebesgue measure.

Chapter V is devoted to quasi-continuous functions. The function $f(p)$ is quasi-continuous on the Lebesgue set I with respect to the determining function $\alpha(T)$ if to every $\epsilon > 0$ one can find a closed set $C_\epsilon \subset I$ with $m_\alpha(I - C_\epsilon) < \epsilon$ such that $f(p)$ be continuous on C_ϵ . The convergence and mean convergence of sequences of quasi-continuous functions are also discussed here. Then chapter VI systematically studies the integration of summable quasi-continuous functions on a Lebesgue set I with respect to $\alpha(T)$. At the end of this chapter the author (using the choice axiom) proves the theorem: If $f(p)$ is summable on I , then there exists a Lebesgue set $J \subset I$ on which $f(p)$ is quasi-continuous and such that $\int_J f(p) d\alpha = \int_I f(p) d\alpha$. In chapter VII the integration of sequences and series of summable functions, also with consideration of mean convergence, is thoroughly discussed. The last section of this chapter is devoted to the study of the Hilbert space H of the functions $f(p)$ which are quasi-continuous and square summable in a Lebesgue set I with respect to a determining function $p(T) \geq 0$. Chapter VIII discusses the reduction formula of integrals, in particular the Fubini theorem. The last section of this chapter brings applications to the construction of complete systems in Cartesian products of finitely many Hilbert spaces H . Finally, chapter IX studies relations between interval functions $\alpha(T)$ and point functions $\varphi(x_1, x_2, \dots, x_r)$ and then discusses Stieltjes integrals with respect to these point functions.

A. Rosenthal (Lafayette, Ind.).

Dell'Agnola, Carlo Alberto. *Intorno ad una generalizzazione del concetto di limite.* Ist. Veneto Sci. Lett. Arti. Ci. Sci. Mat. Nat. 109, 245-260 (1951).

Let Δ be a subdivision of a given interval $[a, b]$ into finitely many subintervals (with norm δ). The author considers functions $\varphi(\Delta)$ of such subdivisions, generalizes the notion of the limit (and also of the upper and lower limit) for these functions $\varphi(\Delta)$, and obtains the usual properties of the limit by means of the usual type of reasoning.

A. Rosenthal (Lafayette, Ind.).

Coelho, Renato Pereira. *Un critère de continuité.* Gaz. Mat., Lisboa 12, no. 50, 27-28 (1951).

Theorem. Let R be a set and let α be a family of subsets of R directed by a relation \leq (so each two elements of α have a common predecessor). Let \mathfrak{F} be the filter of subsets of R determined by the base of subsets $B_A = \bigcup_{A' \leq A} A'$. Let \mathfrak{F}_0 be a filter of subsets of R finer than \mathfrak{F} and such that for each F_0 in \mathfrak{F}_0 there exists A_0 in α such that $A_0 \cap F_0 \neq \emptyset$ if $A \leq A_0$. Let f be a function defined on R with values in a uniform space S . Then $\lim_{\mathfrak{F}} f(x) = L$ if and only if (a) $\lim_{\mathfrak{F}_0} f(x) = L$ and (b) for each neighborhood U in the uniform structure of S there is an A_U in α such that $f(A) \times f(A) \subset U$ if $A \leq A_U$. Corollary. In the plane R with polar coordinates ρ, θ a function f is continuous at the origin if and only if (a') f is continuous at the origin along the half-line $\theta = \theta_0$, and (b') the oscillation of f on the circumference $\rho = r$ tends to zero with r .

M. M. Day.

Pucci, Carlo. *Alcuni teoremi sulle successioni di funzioni di più variabili che possiedono derivate parziali fino all'ordine r .* Ann. Mat. Pura Appl. (4) 31, 129-141 (1950).

The author proves the following theorem on sequences of real functions $f_n(x)$ defined on a domain D of the m -dimen-

sional x -space. Suppose the domain D is open and is such that every pair of its points can be joined by a polygon in D with sides parallel to the axes and length less than a fixed number M . Let the functions $f_n(x)$ be of class $C^{(r)}$ in D , and suppose the partial derivatives of order r of these functions are equicontinuous in D . Then there is a subsequence which converges uniformly on D together with the partial derivatives up to and including those of order r , or else there is a subsequence which diverges at every point of D except possibly at those points lying on an algebraic hypersurface of degree r . Several closely related theorems are included. These results generalize earlier results of Picone [Boll. Un. Mat. Ital. (3) 5, 24-33 (1950); these Rev. 11, 648]. The proof is based on Taylor's expansion with remainder, and on a series of lemmas on sequences of polynomials of degree r .

L. M. Graves (Chicago, Ill.).

Amerio, Luigi. *Un teorema di derivazione per serie e un criterio di eguale continuità ed eguale limitatezza.* Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 14-20 (1950).

The author remarks that the result on term-by-term differentiation obtained by Pucci [see the preceding review] can be secured under the weaker hypothesis that the r th order derivatives are equicontinuous. He also notes that when the functions $f_n(x)$ are bounded at $r+1$ points, and the r th derivatives are equicontinuous, then the functions $f_n(x), f_n'(x), \dots, f_n^{(r)}(x)$ are all equicontinuous and bounded.

L. M. Graves (Chicago, Ill.).

Császár, Ákos. *Sur les nombres de Lipschitz généralisés.* Acta Math. Acad. Sci. Hungar. 1, 277-302 (1950). (French. Russian summary)

A. S. Besicovitch [Math. Z. 30, 514-519 (1929)] defined and studied the Lipschitz numbers of order α ($0 < \alpha < 1$) of $f(x)$, e.g. $L_\alpha^+ f(x) = \limsup_{h \rightarrow 0^+} [f(x+h) - f(x)]/h^\alpha$, and the author [Acta Sci. Math. Szeged 12, Pars B, 211-214 (1950); these Rev. 11, 586] defined and discussed also the approximate Lipschitz numbers, e.g. $\bar{L}_\alpha^+ f(x)$, by using the approximate upper (and lower) limits. Here the author continues and, at the same time, generalizes these studies, replacing h^α by a function $\varphi(h)$ which for $h > 0$ satisfies the following four conditions: (1) $\varphi(h)$ is strictly monotone increasing; (2) $\lim_{h \rightarrow 0^+} \varphi(h) = 0$, $\lim_{h \rightarrow +\infty} \varphi(h) = +\infty$; (3) $\lim_{h \rightarrow 0^+} \varphi(h)/h = +\infty$; (4) $\varphi(h)$ is logarithmically concave, i.e. $\varphi(h) = \exp(\Phi(\log h))$ where $\Phi(t)$, for $-\infty < t < +\infty$, is a concave function (in the wider sense, i.e. linear parts are not excluded). Then the author considers the generalized Lipschitz numbers of $f(x)$, e.g.

$$L_\varphi^+ f(x) = \limsup_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{\varphi(h)},$$

and analogously the generalized approximate Lipschitz numbers of $f(x)$, e.g. $\bar{L}_\varphi^+ f(x)$, using the approximate upper (and lower) limits.

First, theorems are proved which correspond to known theorems concerning L_α^+ etc. Almost everywhere one has $L_\varphi^+ \geq 0$, $L_\varphi^- \geq 0$, $\bar{L}_\varphi^+ \leq 0$, $\bar{L}_\varphi^- \leq 0$, and the same for \bar{L}_φ^+ etc. For every function $f(x)$, the set of points x at which $L_\varphi^- f(x) < L_\varphi^+ f(x) < +\infty$ holds has measure zero. For every measurable function $f(x)$ the analogous theorem concerning generalized approximate Lipschitz numbers holds. Moreover, the author proves the following theorem (which is new even in the special case $\varphi(h) = h^\alpha$): If, for any function $f(x)$, $L_\varphi^+ f(x)$ and $\bar{L}_\varphi^+ f(x)$ are finite on a set E , then $L_\varphi^- f(x)$ and

$\underline{L}_\sigma f(x)$ are also finite almost everywhere on E , and hence $\underline{L}_\sigma^+ f(x) = \underline{L}_\sigma f(x)$, $\underline{L}_\sigma^+ f(x) = \underline{L}_\sigma f(x)$ almost everywhere on E . An analogous theorem is proved, in the case of measurable functions $f(x)$, for the generalized approximate Lipschitz numbers.
A. Rosenthal (Lafayette, Ind.).

Scorza Toso, Annamaria. Un'osservazione sulle funzioni di due variabili continue separatamente rispetto a queste. Rend. Sem. Mat. Univ. Padova 20, 468-469 (1951).

In a rectangle $R: a \leq x \leq b, c \leq y \leq d$, let $f(x, y)$ be finite-valued, measurable with respect to x and continuous with respect to y . According to a theorem of Dragoni, for every $\epsilon > 0$ there exists a closed subset $C_\epsilon(\epsilon)$ of R , such that $f(x, y)$ is continuous on $C_\epsilon(\epsilon)$, and such that the projection of the set $R - C_\epsilon(\epsilon)$ on the x -axis has (linear) measure less than ϵ . The author shows that one can obtain from this theorem in a few lines the following theorem of Bajada: if $f(x, y)$ is continuous in R with respect to x and y separately, then for every $\epsilon > 0$ there exists a closed subset $C(\epsilon)$ of R , such that $f(x, y)$ is continuous on $C(\epsilon)$ and such that the projections of the set $R - C(\epsilon)$ on both the x and the y axis have (linear) measure less than ϵ .
T. Radó (Columbus, Ohio).

Carlson, K. H., and Young, L. C. Continuity of area for harmonic surfaces with boundaries of uniformly bounded length. Proc. Amer. Math. Soc. 3, 88-91 (1952).

Let D be the closed unit disc $|w| \leq 1$ in the $w = u + iv$ plane. If $x(u, v) = [x_1(u, v), x_2(u, v), x_3(u, v)]$ is a vector function whose components are continuous in D and harmonic in the interior of D , then $x(u, v)$ is termed harmonic. The area of the surface determined by $x(u, v)$ is denoted by A . Restricted to the perimeter of D , $x(u, v)$ determines a curve whose length is denoted by L (A and L may be infinite). Let $x^*(u, v)$ be another harmonic vector function, with the associated quantities A^*, L^* . Denote by $d(x, x^*)$ the supremum of $|x(u, v) - x^*(u, v)|$ in D . The authors prove the following result: if x is fixed and $d(x, x^*) \rightarrow 0$, and if there exists a finite constant N such that L^* remains $\leq N$, then $A^* \rightarrow A$. The authors observe that $d(x, x^*)$ is greater than or equal to the Fréchet distance $d_F(x, x^*)$, and hence there arises the (as yet open) question whether the weaker assumptions $L^* \leq N, d_F(x, x^*) \rightarrow 0$ also imply that $A^* \rightarrow A$.
T. Radó (Columbus, Ohio).

Young, Laurent Chisholm. Surfaces paramétriques généralisées. Bull. Soc. Math. France 79, 59-84 (1951).

For the case of Euclidean three-space, the generalized parametric surfaces of L. C. Young may be described as follows. Let $x(u, v)$ be a continuous vector function defined in the unit square $R: 0 \leq u \leq 1, 0 \leq v \leq 1$. In view of the contemplated geometrical interpretation, $x(u, v)$ is termed a "parametric representation". If $x(u, v)$ satisfies a Lipschitz condition, then one speaks of a Lipschitz representation. A Dirichlet representation is defined by the following conditions: (a) For almost every u , x is an absolutely continuous function of v , and for almost every v it is an absolutely continuous function of u . (b) The partial derivatives x_u, x_v have summable squares in R . Let there be chosen any subclass of the Dirichlet representations which contains the Lipschitz representations. The elements of this subclass are termed "elementary representations". Next, let F be the class of all (real-valued) continuous functions $f(x, J)$ of the pair of vectors x, J , satisfying the condition $f(x, tJ) = tf(x, J)$ for $t \geq 0$. Given an elementary representation $x(u, v)$, for every $f \in F$ one has the integral of $f[x(u, v), J(u, v)]$ over R , where

$J(u, v)$ stands for the vector product of x_u and x_v . Let us denote this integral, which depends upon both x and f , by $L(f, x)$. Two elementary representations $x_1(u, v), x_2(u, v)$ are termed equivalent if $L(f, x_1) = L(f, x_2)$ for every $f \in F$. This equivalence relation partitions the class of elementary representations into equivalence classes. Each one of these equivalence classes is then, by definition, an elementary parametric surface S . Thus S may be described by writing $S = [x]$, where this symbol denotes the equivalence class containing the elementary parametric representation $x = x(u, v)$. Accordingly, one can define $L(f, S) = L(f, [x]) = L(f, x)$. In this manner, each elementary surface S determines a functional $L(f, S)$, defined for all $f \in F$. Actually, the basic idea of L. C. Young is to identify S with this functional. Accordingly, a sequence $\{S_n\}$ is termed convergent if the sequence $L(f, S_n)$ is convergent for every $f \in F$. A generalized parametric surface (in the sense of L. C. Young) is now defined as any linear functional $L(f)$, defined for $f \in F$, for which there exists a sequence $\{S_n\}$ of elementary surfaces such that $L(f) = \lim L(f, S_n)$ for every $f \in F$. The purpose of the paper under review is to establish a broad foundation for the general program of developing calculus of variations in terms of these generalized parametric surfaces. Special topics studied in detail include, first, a series of facts which follow more or less directly from the general theory of linear functionals. Next, a preliminary investigation of the concept of the boundary is presented in some detail. Thirdly, a study is made of a type of decomposition of a generalized parametric surface into parts with certain specific properties. Various questions, unsolved or only partially solved, are listed as further objectives in the general program. As a contemplated generalization of L. C. Young's now classical theory of generalized curves, the program initiated in this paper appears as an undertaking of manifest importance and definite promise.
T. Radó (Columbus, Ohio).

Silverman, Edward. An intrinsic property of Lebesgue area. Rivista Mat. Univ. Parma 2, 195-201 (1951).

Let x be a continuous transformation from the square $Q: 0 \leq u, v \leq 1$ into m , the space of bounded sequences. A continuous map f from the interval $a \leq t \leq b$ into Q is called admissible for the pair of points $p, q \in Q$ if $f(a) = p, f(b) = q$. Let $G(x(f))$ denote the ordinary length of the curve (represented by) $x(f)$ and let $\bar{x}_a(p, q) = \inf G(x(f))$ for all admissible f . Then $\bar{x}_a(p, q)$ is the geodesic distance between $x(p)$ and $x(q)$. In case $\bar{x}_a(p, q)$ is continuous on $Q \times Q$, the author defines a continuous transformation x_a from Q into m which satisfies the relation $\|x_a(p) - x_a(q)\| = \bar{x}_a(p, q)$. He then shows that the Lebesgue areas of the surfaces represented by x and x_a are equal. The author points out that, in view of his previous definition for the Lebesgue area of surfaces in metric spaces [same Rivista 2, 47-76 (1951); these Rev. 13, 122], the present result holds when the range of x is any metric space.
R. G. Helsel (Columbus, Ohio).

Silverman, Edward. A note on area. Proc. Amer. Math. Soc. 3, 86-87 (1952).

Let C be the class of continuous transformations x from the square Q into a metric space D ; let \bar{x} be a continuous map from Q into E_1 . Let $\alpha(x)$ be any non-negative functional on C which satisfies the Kolmogoroff principle and agrees with the elementary area of a smooth surface \bar{x} in case x and \bar{x} are isometric. Then $\alpha(x)$ may be regarded as the area of the surface (represented by) x . The author proves that such a functional must show at least one of the following two

peculiarities: (1) $\alpha(x)$ can be finite when range x is a set having positive volume; (2) $\alpha(x)$ can be infinite when range x is a simple arc.
R. G. Helsel (Columbus, Ohio).

Cecconi, Jaurès. *Sull'area di Peano e sulla definizione assiomatica dell'area di una superficie*. Rend. Sem. Mat. Univ. Padova 20, 307-314 (1951).

Let $\phi(S)$ be a functional defined for every Fréchet surface S of the type of the 2-cell and possessing the following properties. a) If $S_n, n=1, 2, \dots$, is a sequence of surfaces converging to S , then the limit inferior of $\phi(S_n)$ is greater than or equal to $\phi(S)$. b) If S is a polyhedral surface then $\phi(S)$ equals the elementary area of S . c) Let (T, Q) be any representation for S upon a unit square Q . If $\pi_i, i=1, \dots, n$, is any finite sequence of pairwise non-overlapping simple polygonal regions in Q , denote by S_i the portion of S with representation (T, π_i) for $i=1, \dots, n$. Then $\phi(S)$ is greater than or equal to $\phi(S_1) + \dots + \phi(S_n)$. d) Let C denote the continuous closed curve which forms the boundary of S . For every plane α let C_α denote the continuous closed curve which is obtained by projecting C upon the plane α . Denote by σ the set of points on α at each of which the topological index relative to the curve C_α is not zero. Then $\phi(S)$ is greater than or equal to the planar measure of σ . Under these conditions the author shows that the functional $\phi(S)$ must coincide with the Lebesgue area of the surface S .

P. U. Reichelderfer (Columbus, Ohio).

Theory of Functions of Complex Variables

✓*Dörrie, Heinrich. *Einführung in die Funktionentheorie*. Verlag von R. Oldenbourg, München, 1951. 559 pp. 48 DM.

The object of this book is to give an account of the classical theory of functions of a complex variable which treats those topics of interest for applications as well as topics which are of interest from a more theoretical point of view. The work assumes a modest background and treats the topics considered in detail. Many examples are studied. The contents of the book follow: complex numbers, point sets, limiting processes; analytic functions, linear fractional transformations, power series (the *Lusin* and *Sierpinski* examples are studied in this section); the *Cauchy* integral theorems and applications; the *Picard* theorem; *lemniscate* functions; algebraic functions; the *Weierstrass* convergence theorem and applications; infinite products; conformal mapping, examples and applications to hydrodynamics; analytic continuation; special functions including the *theta* functions, the *Weierstrass* elliptic functions, the modular functions, the *Jacobian* elliptic functions, the *gamma* function, and the *Riemann* zeta function.

M. H. Heins.

Ryll-Nardzewski, C., et Steinhaus, H. *Sur les séries de Taylor*. Studia Math. 12, 159-165 (1951).

Pólya [Acta Math. 41, 99-118 (1917)] showed that in the space A of functions analytic in the unit circle, those which can be continued form a nondense set, using a particular topology on A . The present paper deals with the same problem from a considerably more general viewpoint. Let A have the topology of pointwise convergence, let X be an arbitrary Banach space and let T be a linear transformation of X into A . The authors then show that there is an open set G of the circumference $C, |z|=1$, such that each function

$T(x)$ is regular on G , but that there is a set $S \subset X$ of first category such that every point off G is singular for $T(x)$ when $x \in S$. In particular, if X is extensive enough so that G is void, then $T(x)$ has C as a cut for each $x \in S$. Choose X as the space of sequences $x = \{a_n\}$ where $\|x\| = \sup_n |a_n| n^{1/n} < \infty$, with $T(x)(z) = \sum a_n z^n$. The functions $T(x)$ in A have radial boundary values on C which are infinitely differentiable, but those which can be continued arise from a set of first category in X .
R. C. Buck (Madison, Wis.).

Belardinelli, G. *Sulla convergenza assoluta delle serie di fattoriali*. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 13(82), 239-254 (1949).

The author establishes two theorems concerning the convergence of factorial series $\sum c_n/p_n(z)$, $p_n(z) = \prod_{k=1}^n (z - a_k)$, $a_k \rightarrow \infty$. (I) If $\sum |a_k|^{-1}$ converges and there is a point at which the factorial series converges absolutely, then $\sum |c_n/p_n(z)|$ converges uniformly in every closed region not containing the points a_k . (II) Suppose $\sum |a_k|^{-1}$ diverges and $|\sum |a_k|^{-1}| > c \sum |a_k|^{-1}$ for some constant $c, n=1, 2, \dots$. Set $A = \liminf_{n \rightarrow \infty} \arg \sum |a_k|^{-1}$, $B = \limsup_{n \rightarrow \infty} \arg \sum |a_k|^{-1}$ and assume $B - A < \pi$. Then the series $\sum |c_n/p_n(z)|$ converges uniformly in any closed region not containing the points a_k and contained in the angle $\pi/2 - A < \theta < \pi/2 - B$. A number of corollaries of (II) are proved which examine admissible patterns for the arguments of the a_k . It should be noted that these theorems on factorial series are closely allied to similar results obtained recently by Y. Martin [Bull. Sci. Math. (2) 75, 21-32 (1951); these Rev. 12, 813] for series of faculties.
E. N. Nilson (Hartford, Conn.).

Martin, Yves. *Sur les dérivées successives de certaines fonctions analytiques*. Bull. Sci. Math. (2) 75, 166-171 (1951).

This paper contains the proof of theorems related to a recent result of Rådström [Proc. Nat. Acad. Sci. U. S. A. 35, 399-404 (1949); these Rev. 11, 22] some of which the author announced in a previous paper [C. R. Acad. Sci. Paris 232, 136-137 (1951); these Rev. 12, 688]. If f is analytic at 0 let $S(f)$ be the set of points which are limit points of f, f', f'', \dots , and let Z be those f for which $S(f)$ contains the origin. Let $a \neq 1$. The author considers the class of functions $f(z) = \sum b_n z^n/n!$ which can be modified by multiplying the coefficients by a sequence $\{w_n\}$ composed only of the numbers 1 and a , so as to belong to Z . He proves that a necessary and sufficient condition for this is that $\limsup |b_{n+1}/b_n| = +\infty$. This contains the results of both Rådström and of Erdős [Proc. Amer. Math. Soc. 2, 205-206 (1951); these Rev. 12, 815], who discussed the case of $f(z)$ entire, and $|w_n| = 1$. The author also discusses the nature of $S(g)$ for g a $\{w_n\}$ -modification of f , when

$$\limsup |b_{n+1}/b_n| < \infty.$$

R. C. Buck (Madison, Wis.).

Meiman, N. *Some comparison theorems for analytic functions*. Doklady Akad. Nauk SSSR (N.S.) 82, 185-188 (1952). (Russian)

The author introduces the following definitions. Class E consists of functions $F(z)$ regular in the whole plane except for isolated singular points, satisfying $|F(z)/F(z)| < 1$ for $y < 0$. Writing $f(z) = g(z) + ih(z)$ is to imply that $g(z)$ and $h(z)$ are real on the real axis, and then $\tilde{f}(z)$ means $g(z) - ih(z)$. If $f(z)$ is regular in the whole plane except for isolated singular points, and $F(z)$ has the same property in the lower half plane, then $f(z)$ is called subordinate to $F(z)$ if

$|f(z)/F(z)| \leq 1$ and $|\bar{f}(z)/\bar{F}(z)| \leq 1$ for $y < 0$. Subclasses of E are M (meromorphic functions in E), G (entire functions in E), HB (entire functions free of zeros in $y \leq 0$), and various subclasses of HB . With the aid of theorems from a previous paper [same Doklady (N.S.) 71, 609-612 (1950); these Rev. 11, 509], the author proves the following theorems. 1. If $F(z) \in E$ and $f(z)$ is subordinate to $F(z)$, then for $|t| < 1$, $F_t(z) = F(z) - t f(z)$ belongs to E . 2. Under the hypotheses of Theorem 1, $|f'(x)| \leq |F'(x)|$ on the real axis, and

$$|FF'| \sin \beta + |ff'| \sin \gamma \geq |\bar{F}\bar{F}' - \bar{f}\bar{f}'|,$$

where

$$(k \cos \alpha \cos \gamma - \cos \beta)^2 \leq (1 - k^2) \sin^2 \alpha, \quad k = |f'(x)/F'(x)|, \\ \cos \alpha = |f(x)/F(x)|, \quad \beta = \arg F'(x) - \arg F(x), \\ \gamma = \arg f'(x) - \arg f(x).$$

Applications to entire functions of order 1 and 2 are pointed out. 3. Let $F_t(z)$ (of Theorem 1) belong to E for all $|t| < 1$; then $|f(z)| \leq |F(z)|$ for $y < 0$. 4. Let A be a linear operator from E to E , such that $A\bar{f} = \overline{Af}$, and f subordinate to $F \in E$. Then A^*f is subordinate to A^*F ; and E can be replaced by the subclasses mentioned above.

The author discusses the connections between his results and those of B. Levin [Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 45-84 (1950); these Rev. 11, 510]. He also notes that the general results of Wigner [Ann. of Math. (2) 53, 36-67 (1951); these Rev. 12, 490] are contained in older results of Grommer, Čebotar'ev and Melman, with references in particular to Melman [C. R. (Doklady) Acad. Sci. URSS (N.S.) 40, 46-49, 179-181 (1943); these Rev. 6, 59] and Čebotar'ev and Melman [Trudy Mat. Inst. Steklov 26 (1949); these Rev. 11, 509].

R. P. Boas, Jr.

Sasaki, Yasuharu. Theorems on the convexity of bounded functions. Proc. Japan Acad. 27, 122-129 (1951).

Let $F(z)$ be regular in $|z| < 1$ with $|F(z)| < M$, $F(0) = 0$, $F'(0) = 1$. It is shown that $F(z)$ is schlicht and convex for $|z| < R \leq 1$ where

$$R^3 - 3MR^2 + (4M^2 - 1)R - M = 0.$$

The bound is sharp. If, in addition, $F(z)$ is known to be schlicht in $|z| < 1$, then $R = \frac{1}{2}[t - \sqrt{(t^2 - 4)}]$ where $t (\geq 2)$ satisfies

$$t^3 - 2\left(1 + \frac{2}{M}\right)t^2 - 4\left(2 - \frac{6}{M} + \frac{1}{M^2}\right)t - \frac{8}{M^2} = 0.$$

The bound is again sharp. $R \rightarrow 2 - \sqrt{3}$ as $M \rightarrow +\infty$.

M. S. Robertson (New Brunswick, N. J.).

Bernardi, S. D. Two theorems on schlicht functions. Duke Math. J. 19, 5-21 (1952).

Let S be the class of all $f(z) = z + a_2 z^2 + \dots$, regular and schlicht in $|z| < 1$. The estimate $|a_4| < 4.0891$ is established. Also all functions of S are determined whose coefficients are algebraic integers in the extension $K(\sqrt[N]{N})$ over the rational field, where $N \geq 1$ is a square-free integer. There are 17 such functions for $N = 1$, 7 for $N = 2$, 27 for $N = 3$ and 5 for $N > 3$. [Cf. B. Friedman, Duke Math. J. 13, 171-177 (1946); these Rev. 8, 22.] W. W. Rogosinski (Newcastle-upon-Tyne).

***Goluzin, G. M. On subordinate univalent functions.** Trudy Mat. Inst. Steklov., v. 38, pp. 68-71. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

By integrating a Loewner type differential equation, the author proves the following theorem. Let $f_1(z)$ and $f_2(z)$

be regular and univalent in $|z| < 1$, and suppose that $f_1(0) = f_2(0)$, $\arg f_1'(0) = \arg f_2'(0)$ and $B_1 \subset B_2$, where B_j is the image domain of $|z| < 1$ under $f_j(z)$. Then for $|z| < 3 - 2\sqrt{2}$, $|f_1'(z)| \leq |f_2'(z)|$ with equality only if $f_1(z) = f_2(z)$. For any z outside this circle, there exist functions satisfying the hypotheses for which $|f_1'(z)| > |f_2'(z)|$. The author asserts without proof, that if in addition $f_1(z)$ and $f_2(z)$ have p -fold symmetry then the constant $3 - 2\sqrt{2}$ is replaced by $((p+1)^{1/2} - p^{1/2})^{2/p}$. A. W. Goodman.

Wigner, E. P. On the connection between the distribution of poles and residues for an R function and its invariant derivative. Ann. of Math. (2) 55, 7-18 (1952).

The class R of analytic functions, considered by the author in an earlier paper [Ann. of Math. 53, 36-67 (1951); these Rev. 12, 490], consists of functions $f(z)$ which are meromorphic and satisfy $\text{Im } \{f(z)\} > 0$ and $\text{Im } \{f(z)\} < 0$ for $\text{Im } \{z\} > 0$ and $\text{Im } \{z\} < 0$, respectively. The functions $f(z)$ can be shown to be of the form

$$(*) \quad f(z) = c + \sum \frac{\alpha_r}{z - z_r},$$

where c, z_r are real and $\alpha_r > 0$. The function $f'(z)[1 + f^2(z)]^{-1}$ is called by the author the invariant derivative of $f(z)$ and is shown to have an expansion of the type

$$(**) \quad c' + \sum \frac{\beta_r}{(z - x_r)^2 + \beta_r^2}$$

with real β_r, x_r and $c' \geq 0$.

Statistical R -functions are functions of the above type for which the constants z_r and α_r in (*), or the constants β_r and x_r in (**), show a statistical distribution. The problem discussed in the present paper and solved in a special case is to determine the distributions of the α_r and z_r if those of the β_r and x_r are given. Z. Nehari (St. Louis, Mo.).

Davis, Philip. An application of doubly orthogonal functions to a problem of approximation in two regions. Trans. Amer. Math. Soc. 72, 104-137 (1952).

Let G and B be bounded regions, $G \subset B$. The problem of approximating a function $f(z)$, analytic on G but not throughout B , on G by functions $f_M(z)$ analytic on B ($|f_M(z)| \leq M$; M given) was studied in a number of papers by Walsh; and by Walsh and Nilson [e.g., Trans. Amer. Math. Soc. 65, 239-258 (1949); these Rev. 10, 524]. In that paper the norm of $f(z)$ on G is defined by integral means, of index p ($p > 0$), on the boundary of G . The author continues this work, but uses surface integrals. He takes $p = 2$ and introduces the sets L_G^2, L_B^2 of functions which are analytic on G or B and of finite norm $\|f\|_G$ or $\|f\|_B$, respectively, where $\|f\|_G = (\int_G |f|^2 dA)^{1/2}$, etc. He deals with the problem: given $f(z) \in L_G^2$, to find $f_M(z) \in L_B^2$ such that $\|f_M\|_B \leq M$ and that $\|f - f_M\|_G$ is minimized by f_M . His method is based on S. Bergman's theory [The kernel function . . . , Amer. Math. Soc., New York, 1950; these Rev. 12, 402] of functions $\phi_n(z)$ ($n = 0, 1, 2, \dots$) which are simultaneously orthogonal over G and B ,

$$\iint_G \phi_m(z) \phi_n(z) \bar{z} dz = \delta_{m,n}; \quad \iint_B \phi_m(z) \phi_n(z) \bar{z} dz = k_n \delta_{m,n} \\ (k_n > 1, \sum k_n^{-1} < \infty)$$

where $\phi_n(z)$ is the complex conjugate of $\phi_n(z)$, and on the Fredholm theory of integral equations. The system $\{\phi_n(z)\}$, certainly closed with respect to L_B^2 , is required to be closed in L_G^2 also, a property which depends on the nature of the

boundaries of G and B . Let $K_B(z, w) = \sum \phi_n(z) \phi_n(w) / k_n$ be the kernel function of B , Tg the transformation

$$\int \int_G K_B(z, w) g(w) dt_w$$

of $g(z) \in L_G^2$. The author obtains explicit representations of $f_M(z)$, etc. (as was to be expected, since $p=2$; cf. Walsh and Nilson, loc. cit., §4).

Theorem 4. Let $M > 0$, $f(z) \in L_G^2$ and therefore of the form $f(z) = \sum a_n \phi_n(z) (\sum |a_n|^2 < \infty)$. Then there is a unique solution of the approximation problem,

$$f_M(z) = f(z, \lambda) = \sum a_n \phi_n(z) / (1 + \lambda k_n)$$

where $\lambda = \lambda(M)$ is the positive root of the equation $M^2 = \sum |a_n|^2 k_n (1 + \lambda k_n)^{-2}$. Again the measure of approximation is

$$\|f - f_M\|_{G^2}^2 = \lambda^2 \sum |a_n|^2 k_n^2 (1 + \lambda k_n)^{-2},$$

and $f_M \rightarrow f$ ($M \rightarrow \infty$) uniformly in any closed subregion of G . Theorem 5. Let $k^*(z, w; \lambda)$ be the resolvent kernel of the integral equation $f(z) = g(z) + \lambda^{-1} Tg$. If $f \in L_G^2$ then

$$f(z, \lambda) = \int \int_G k^*(z, w; \lambda) f(w) dt_w$$

is, for each $\lambda > 0$, in L_B^2 and of best approximation in G to $f(z)$ (and so on). The convergence of $f_M(z)$ to $f(z)$ is interpreted as a regular method of summation by integral means (Theorem 6). The totality of the functions of best approximation is investigated, i.e. of the $F(z) \in L_B^2$ such that there exists a $M > 0$ and a $f(z) \in L_G^2$ for which $F(z)$ is the $f_M(z)$. That is the case if and only if $F(z) = Tg$ for some $g(z) \in L_G^2$, or if $F(z) = \sum a_n \phi_n(z)$ with $\sum |a_n|^2 k_n^2 < \infty$ (Theorem 7). A number of the identities related to the integral equations are interpreted as theorems on best approximation. Finally, special cases and generalizations of the problem are discussed.

H. Kober (Birmingham).

- ✓ *Carathéodory, C. Conformal representation. 2d ed. Cambridge Tracts in Mathematics and Mathematical Physics, no. 28. Cambridge, at the University Press, 1952. x+115 pp. \$2.50.

Photo-offset reprint of the 1932 edition with some slight changes and the addition of a new chapter, The general theorem of uniformisation.

- Komatu, Yūsaku, and Ozawa, Mitsuru. Conformal mapping of multiply connected domains. I. Kodai Math. Sem. Rep. 1951, 81-95 (1951).

The authors study the conformal mapping of multiply-connected regions of finite connectivity onto certain types of canonical regions (plane slit along horizontal and vertical segments, plane slit along radial segments and circular arcs, parallel strip slit along perpendicular segments). Typical of the results is the following theorem: Any region of connectivity n bounded by n continua admits a one-to-one conformal mapping onto the extended plane slit along horizontal and vertical segments, the mapping being such that p assigned boundary components correspond to vertical slits and the remainder to horizontal slits. The mapping subject to certain normalization conditions is unique. The proofs are based upon extremal methods.

M. Heins.

- Komatu, Yūsaku. On conformal slit mapping of multiply-connected domains. Proc. Japan Acad. 26, no. 7, 26-31 (1950).

This paper is concerned with a generalization to the case of multiply-connected domains of Loewner's differential

equation for univalent slit-mappings of the unit circle. The canonical domain D chosen is the circular ring with concentric circular slits, further normalized by taking the unit circumference as the outer boundary. The author considers conformal mappings $w = f(z)$ of D onto a domain D' of the same type to which a slit joining a point on $|w|=1$ and an interior point has been applied, and he connects the mappings $w = z$ and $w = f(z)$ by a one-parameter family $w = f(z, q)$ corresponding to the growth of the slit from a point on $|w|=1$ to its final shape; the parameter q which proves most convenient for this purpose is the radius of the inner boundary of D . The partial differential equation for $f(z, q)$ which is finally obtained involves the Green's function and the harmonic measures of D and also the functional dependence on q of the radii of the circular slit components of the boundary of D .

Z. Nehari (St. Louis, Mo.).

- Meschkowski, Herbert. Beziehungen zwischen den Normalabbildungsfunktionen der Theorie der konformen Abbildung. Math. Z. 55, 114-124 (1951).

This paper is concerned with the derivation of a number of relations between some of the canonical conformal mappings of a multiply-connected domain. The principal tool used is the contour integration method, first used for a similar purpose by Grunsky [Schr. Math. Sem. u. Inst. Angew. Math. Univ. Berlin 1, 95-140 (1932)] and systematically exploited by Garabedian and Schiffer [Trans. Amer. Math. Soc. 65, 187-238 (1949); these Rev. 10, 522]. Employing suitable combinations of pairs of conjugate differentials, the author arrives at a number of identities between some of the fundamental domain functions. A somewhat different technique, involving repeated inversions, is used for deriving a representation of the function mapping a doubly-connected domain onto the full plane minus two closed circular discs.

Z. Nehari (St. Louis, Mo.).

- Hervé, Michel. Quelques propriétés des transformations intérieures d'un domaine borné. Ann. Sci. École Norm. Sup. (3) 68, 125-168 (1951).

This paper supplies detailed proofs to results announced by the author in three notes [C. R. Acad. Sci. Paris 230, 609-610, 707-708, 1491-1493 (1950); these Rev. 11, 650, 589, 719]. The interior transformations mentioned in the title are, in the case of one complex variable, mappings of a domain D by means of an analytic function $f(z)$ which is regular in D and takes there values likewise situated in D . The author also considers the generalization of this concept to the case of mappings effected by two analytic functions of two complex variables.

The central problem treated in the case of one complex variable is the determination of the so-called "fixed point constant" $\Omega(D, f) = \max |f'(z)|$, where $z \in D$ and $f(z)$ ranges over the functions yielding interior transformations of D in the above sense. In the case in which D is not simply-connected (the only case of interest), the method employed by the author is based on the appropriate use of the mapping of the unit disc onto the universal covering surface of D . Accordingly, the method is mainly applicable to the family of functions $f(z)$ which are regular but not necessarily single-valued in D . The discussion of the single-valued case is based on Grunsky's generalization of the Schwarz lemma to multiply-connected domains [Jber. Deutsch. Math. Verein. 52, 118-132 (1942); these Rev. 4, 270].

Z. Nehari (St. Louis, Mo.).

Garabedian, P. R. A partial differential equation arising in conformal mapping. *Pacific J. Math.* 1, 485-524 (1951).

Let D be a plane domain and let $L^2(D)$ be the class of analytic functions $f(z)$ of the complex variable z which are regular and single-valued in D and satisfy $\iint_D |f(z)|^2 dx dy < \infty$. It was shown by Schiffer [*Duke Math. J.* 13, 529-540 (1946); these *Rev.* 8, 371] that the Bergman kernel function $K(z, t)$ of the class $L^2(D)$ associated with the norm $\iint_D |f(z)|^2 dx dy$ is related to the harmonic Green's function of D by means of the formula

$$(*) \quad K(z, t) = -\frac{2}{\pi} \frac{\partial^2 g(z, t)}{\partial z \partial \bar{t}}.$$

The author shows that similar relations hold for the kernels $K_\rho(z, t)$ associated with the norm $\iint_D \rho(z) |f(z)|^2 dx dy$, where $\rho(z)$ is a given positive and continuously differentiable function in the closure of D . If $G(z, t)$ is the Green's function of D for the differential equation

$$\frac{\partial}{\partial z} \left[\frac{1}{\rho(z)} \frac{\partial u}{\partial \bar{z}} \right] = 0,$$

the formula generalizing (*) is found to be

$$K_\rho(z, t) = -\frac{2}{\pi \rho(z) \rho(t)} \frac{\partial^2 G(z, t)}{\partial z \partial \bar{t}}.$$

Using a method of proof developed in an earlier joint paper with Schiffer [*Ann. of Math.* (2) 52, 164-187 (1950); these *Rev.* 12, 89], the author not only establishes the formal validity of this relation but also gives at the same time an existence proof for the Green's function in question.

A similar discussion is carried out for the kernels associated with the norm $\iint_D \rho(z) u^2(z) dx dy$, where $u(z)$ is a real harmonic function in D . In the case in which $\rho=1$, the kernel is found to be related to an extremal problem considered by Friedrichs [*Trans. Amer. Math. Soc.* 41, 321-364 (1937)]. Some numerical examples are given.

Z. Nehari (St. Louis, Mo.).

Myrberg, P. J. Sur les fonctions automorphes. *Ann. Sci. École Norm. Sup.* (3) 68, 383-424 (1951).

The present paper is a comprehensive study of automorphic groups and functions. The following is a summary of the contents. 1) Survey of classical results. 2) Analytic representation of Fuchsian functions. Here the possibility of exploiting functions generalizing the sigma functions of the theory of elliptic functions for Fuchsian groups is considered, the hyperelliptic case being examined in detail. 3) The classification of Fuchsoid groups (for the case of genus zero) with the aid of the notion of capacity (and related concepts) and the relation of this classification to the problem of the analytic representation of Fuchsoid functions. 4) Function theory on a Riemann surface. 5) Automorphic functions of several varieties.

M. H. Heins (Paris).

Sario, Leo. A linear operator method on arbitrary Riemann surfaces. *Trans. Amer. Math. Soc.* 72, 281-295 (1952).

An operator method is presented for a unified treatment of mapping, existence, uniqueness, and boundary value problems on Riemann surfaces. Linear operators are considered which associate with every function v harmonic in the neighborhood of the relative boundary of a suitably restricted region S of a Riemann surface R a unique harmonic function Lv on S . Given a finite family l of closed

curves on R , a "singular function" (real) s is selected on S which is harmonic near l and admits harmonic continuation across l . For s suitably chosen, there exists on R a real function p harmonic in a neighborhood of l and such that $p-s=L(p-s)$ on S . The function p is unique. By suitably choosing s and L , the corresponding p yield solutions for mapping and existence problems. Numerous applications are given. The present investigation finds its starting point in the Schwarz alternating method.

M. Heins.

Yôjôbô, Zuiman. On the Riemann surfaces, no Green function of which exists. *Math. Japonicae* 2, 61-68 (1951).

The author establishes an extension of the Phragmén-Lindelöf lemma for regions of a parabolic Riemann surface. Applications to covering properties of conformal mappings of such surfaces into the extended plane and to an extension of the Gross-Tsuji theorem.

M. Heins.

Kuroda, Tadashi. On the type of an open Riemann surface. *Proc. Japan Acad.* 27, 57-60 (1951).

Proofs of theorems of Laasonen [*Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys.* no. 11 (1942); these *Rev.* 8, 24] and Ahlfors [*C. R. Acad. Sci. Paris* 201, 30-32 (1935)] concerning the type of a Riemann surface.

M. Heins.

Kuroda, Tadashi. Notes on an open Riemann surface. *Kôdai Math. Sem. Rep.* 1951, 61-63 (1951).

Remarks on the maximum principle for harmonic functions on non-compact Riemann surfaces.

M. Heins.

Mori, Akira. On the existence of harmonic functions on a Riemann surface. *J. Fac. Sci. Univ. Tokyo. Sect. I.* 6, 247-257 (1951).

The author is concerned with the classes of Riemann surfaces (B_0) (not tolerating non-constant bounded harmonic functions) and (D_0) (not tolerating non-constant harmonic functions with bounded Dirichlet integral). Theorems of Bader and Parreau [*C. R. Acad. Sci. Paris* 232, 138-139 (1951); these *Rev.* 12, 603] are established. These theorems lead to a maximum principle whose consequences are further explored.

M. Heins (Paris).

Parreau, Michel. Fonctions harmoniques et classification des surfaces de Riemann. *C. R. Acad. Sci. Paris* 234, 286-288 (1952).

Harmonic functions u on a Riemann surface which satisfy the condition that $\varphi(|u|)$ admits a harmonic majorant for φ increasing and convex are studied. If $\lim_{t \rightarrow +\infty} \varphi(t)/t = +\infty$, the existence of a non-constant u having the above property is equivalent to the existence of a non-constant bounded harmonic function. The concept of quasi-bounded harmonic functions is introduced and the above result is related to the Martin theory of the representation of positive harmonic functions.

M. Heins (Paris).

Schneider, Theodor. Zur Charakterisierung der algebraischen und der rationalen Funktionen durch ihre Funktionswerte. *Acta Math.* 86, 57-70 (1951).

In a recent paper [K. Dörge, *Math. Ann.* 122, 259-275 (1950); these *Rev.* 12, 688] criteria were established that a Puiseux series represent an algebraic function of a complex variable. These criteria involved the algebraic character both of the values assumed by the function on a set of points and of the coefficients of the Puiseux series. The present author obtains criteria for the algebraic or rational character

of a function solely in terms of the values assumed by the function on a sequence of points. More specifically, let $g(z)$ be regular at $z=0$ (it may be multiple-valued). Then, $g(z)$ is an algebraic function of z if, and only if, a) the values assumed by $g(z)$ on a sheet of its Riemann surface over those points $z=1/\nu$, $\nu=1, 2, \dots$, which lie in a fixed neighborhood of $z=0$, are algebraic numbers of at most degree s , where s is a fixed natural number, and

$$b) \limsup_{\nu \rightarrow \infty} \frac{\log \log H(g(1/\nu))}{\log \nu} < 1,$$

where $H(g(1/\nu))$ denotes the "height" of $g(1/\nu)$, i.e., the maximum of the moduli of the relatively prime integral coefficients of the irreducible equation for $g(1/\nu)$.

Similarly, $g(z)$ is a rational function of z if, and only if, a) the values assumed by $g(z)$ on a sheet of its Riemann surface over those points $z=1/\nu$, $\nu=1, 2, \dots$, which lie in a fixed neighborhood of $z=0$, belong to a fixed algebraic number field, and

$$b) \limsup_{\nu \rightarrow \infty} \frac{\log \log H(g(1/\nu))}{\log \nu} < 1.$$

It turns out that in both criteria conditions a) and b) together imply the sharper condition

$$\limsup_{\nu \rightarrow \infty} \frac{\log H(g(1/\nu))}{\log \nu} < t,$$

t a fixed number. Analogous criteria are established which involve this sharper condition, but on the other hand, less stringent conditions on the algebraic character of the numbers $g(1/\nu)$.
W. Seidel (Rochester, N. Y.).

Röhrli, Helmut. *Funktionenklassen auf geschlossenen Riemannschen Flächen*. Math. Nachr. 6, 355-384 (1952).

Suppose that \mathfrak{F} is a closed Riemann surface with the function field $C(w, z)$, C the field of all complex numbers. Then each abelian integral $V = \int (w, z) dz$ with $(w, z) \in C(w, z)$ and variable upper limit on \mathfrak{F} determines a class k which consists of all functions $r(w, z) \exp V$, $r(w, z) \in C(w, z)$ to be considered on the canonically dissected simply connected surface \mathfrak{F}^* belonging to \mathfrak{F} . Let p denote a point of \mathfrak{F}^* with the local uniformizing variable t , then $V = A \log t + \sum_{\lambda=1}^{\infty} a_{\lambda} t^{\lambda}$ defines a complex number α with $A = \alpha \bmod 1$, $0 \leq R(\alpha) < 1$, and a function $f_p(t) = \exp \{ \alpha \log t + \sum_{\lambda=1}^{\infty} a_{\lambda} t^{\lambda} \}$ which are called the "reduced exponent" and the "characterizing function" of the class k at p , respectively. The class k is a $C(w, z)$ -module which is closed under differentiation with respect to z . The author investigates the arithmetic properties of the functions of k for arbitrary V and thus generalizes results of Prym, Rost and R. König. The order m of $y \in k$ at p is defined by $y/f_p(t) = t^m p(t)$ with $p(0) \neq 0$. This definition enables the author to carry over, with some changes, a good portion of the results of the classical theory of algebraic functions which center around the theorem of Riemann-Roch. The ideal-theoretic method of Hensel and Landsberg is used. The complementary class \bar{k} of k is the class which is defined by W^{-1} . Then the residue theorem, for example, takes the form $\sum \text{Res } y d\bar{w}$ where $d\bar{w}$ is a differential of the complementary class \bar{k} , i.e. rk if $d\bar{w} = r dz$. Various forms for the generalized Riemann-Roch theorem are given which finally lead to a theory of Weierstrass points of a class.

O. F. G. Schilling (Chicago, Ill.).

Garabedian, P. R. A Green's function in the theory of functions of several complex variables. Ann. of Math. (2) 55, 19-33 (1952).

Let D be a region bounded by smooth hypersurfaces C in the $2k$ -dimensional Euclidean space of k complex variables z_1, \dots, z_k , and let $\alpha_1, \dots, \alpha_k$ be suitable complex direction cosines on C . Given an arbitrary complex-valued function p on C , the author studies the extremal problem $\int_D |\Delta \beta|^2 d\tau = \min$, where Δ is the $2k$ -dimensional Laplacian operator and β is a function defined in D which satisfies the boundary conditions $\sum \alpha_i \partial \beta / \partial \bar{z}_i = p$. This problem is shown to play a role analogous to the Dirichlet problem in the case of one complex variable. A Green's function is introduced in terms of which the solution can be expressed as a boundary integral, and a relation between this Green's function and the Bergman kernel function for several complex variables is obtained. Similar results are derived for the Szegő kernel function in several complex variables. Questions of existence are not discussed.

Z. Nehari (St. Louis, Mo.).

Hitotumatu, Sin. On the possibility of the Weil's integral representation. Proc. Japan Acad. 27, 279-281 (1951).

L'auteur observe que la théorie des idéaux de fonctions analytiques de K. Oka et H. Cartan permet de prouver facilement l'existence du noyau utilisé par A. Weil dans sa classique formule intégrale [Math. Ann. 111, 178-182 (1935)].

H. Cartan (Paris).

Hersch, Joseph, et Pfluger, Albert. Généralisation du lemme de Schwarz et du principe de la mesure harmonique pour les fonctions pseudo-analytiques. C. R. Acad. Sci. Paris 234, 43-45 (1952).

A (complex-valued) map of a region G of the complex plane is termed D -pseudo-analytic by the authors provided that (1) it is interior in the sense of Stoilow, (2) a module M of each "quadrilateral" (= Jordan region with four distinguished boundary points) and a module M' of its image satisfy $D^{-1}M \subseteq M' \subseteq DM$ ($D \geq 1$); by a module of a quadrilateral is meant one of the ratios of the sides of a rectangle obtained from the quadrilateral by a conformal mapping which carries the distinguished points into the vertices. Generalizations of function-theoretic results (Schwarz lemma, Jensen inequality, harmonic measure theorems) are stated for D -pseudo-analytic functions. It is indicated that the proofs are based on the inequalities satisfied by the modules M, M' of a quadrilateral and its image and extremal properties of the modules.

M. Heins.

Potapov, V. P. On holomorphic matrix functions bounded in the unit circle. Doklady Akad. Nauk SSSR (N.S.) 72, 849-852 (1950). (Russian)

The author proves a series of theorems on matrix-functions $w(\zeta)$ of a complex scalar variable ζ , culminating in the following analogue of the Poisson-Jensen formula: Let $w(\zeta)$ be holomorphic and bounded in $|\zeta| < 1$, and let its determinant not vanish identically. Then there exists a representation

$$w(\zeta) = B(\zeta) \int_0^1 \exp \left[\frac{\lambda + e^{i\theta(t)}}{\lambda - e^{i\theta(t)}} dE(t) \right],$$

where $B(\zeta)$ is a "Blaschke product" of certain canonical factors associated with the zeros of the determinant of $w(\zeta)$, $E(t)$ is a monotone increasing family of Hermitian matrices, $t = \text{tr } E(t)$ and $\theta(t)$ is a monotone increasing function with

$0 \leq \theta(t) \leq 2\pi$. The integral \hat{f} is a Stieltjes multiplicative integral. There are slight affinities with the work of C. L. Siegel [Amer. Math. J. 65, 1-86 (1943); these Rev. 4, 242], and with the theory of systems of differential equations [see Livšic, same Doklady 72, 1013-1016 (1950); these Rev. 13, 747].

F. V. Atkinson (Ibadan).

Theory of Series

✓Dörrie, Heinrich. *Unendliche Reihen*. Verlag von R. Oldenbourg, München, 1951. xi+725 pp. 58 DM.

The first part, pp. 1-172, is a resumé of fundamental ideas involving sets, sequences, and functions. The second and main part treats convergence of series, pp. 174-271; double series, pp. 272-292; power series pp. 293-379; infinite products, pp. 380-442; trigonometric series, pp. 443-497; Dirichlet series, pp. 498-531; asymptotic series, pp. 532-558; and Bessel functions and series, pp. 559-641. The third and last part, pp. 645-720, gives brief applications in number theory, geometry, and physics.

This is not a modern treatise on the modern theory of series. It is, despite some faults, a very good old-fashioned book. It should have been published in 1905 when it would have been the leading book in its field; then its relative value would have steadily decreased as more modern books came along, but its present value would still be far above zero. It is perhaps the most extensive single exposition of the theory of series as it existed before Riemann integrals and convergence were forced to share the limelight with Lebesgue integrals and with other methods for evaluation of series. There are no problems and examples other than those worked out in the text. Among the parts of the book that should be most attractive to students are the extensive chapters on trigonometric and Bessel series, and several elementary treatments involving geometry, number theory, Euler products, theta series, and Dirichlet L -series.

Many remarks such as the following could be made. The chapter on double series is quite obsolete, containing no mention of the Pringsheim concept of convergence. In treating the Gibbs phenomenon in the theory of Fourier series, the author does not follow the practice of copying absurd values of the fundamental constants from other books; he gives good approximations to the values of the constants k and K defined by $\int_0^\pi u^{-1} \sin u du = k = \frac{1}{2}K\pi$. Expositions of this subject should give a reference to an interesting account [Math. Tables and other Aids to Computation 2, 195-196 (1946)] where the values $k = 1.851937051982466$ and $K = 1.178979744472167270232029$ are given by D. H. Lehmer. However, the author strives to perpetuate the incorrect belief that the phenomenon was first discovered by Gibbs in 1898. The phenomenon was discovered 50 years earlier by Henry Wilbraham [Cambridge and Dublin Math. J. 3, 198-201 (1848)]. Referring to what J. J. Sylvester [Amer. J. Math. 5, 251-330 (1882), p. 260] calls "Dr. F. Franklin's remarkable method of proving Euler's celebrated expansion of $(1-x)(1-x^2)(1-x^3)\dots$ " and citing Franklin's publication [C. R. Acad. Sci. Paris 92, 448-450 (1881)], the author follows the C. R. in erroneously crediting the paper to "J. Franklin". The paper was written by Fabian Franklin (1853-1939).

R. P. Agnew (Ithaca, N. Y.).

*Szász, Otto. *Introduction to the theory of divergent series*. Revised ed. Department of Mathematics, Graduate School of Arts and Sciences, University of Cincinnati, Cincinnati, Ohio, 1952. v+81 pp. \$2.75.

Except for minor additions inserted at the ends of sections and lengthening the booklet by 9 pages, this is an exact reproduction of the first edition [1944; these Rev. 6, 45].

R. P. Agnew (Ithaca, N. Y.).

Knödel, W., und Schmetterer, L. *Über ein Problem von Herrn Leja betreffend im Mittel monotone Folgen*. Publ. Math. Debrecen 2, 121-133 (1951).

Following an idea of F. Leja [Ann. Soc. Polon. Math. 19, 133-139 (1947); these Rev. 9, 85] the authors consider real sequences $\{a_n\}$ for which

$$(1) \quad \frac{a_{n+1} + \dots + a_{n+p}}{p} \geq \frac{a_{n+q+1} + \dots + a_{n+p+q}}{q} \quad (n=0, 1, 2, \dots),$$

where p, q are fixed positive integers. Leja proved that, if $q=1$, then a_n tends to a finite limit or to $-\infty$; and he raised the question whether a_n need tend to a limit (finite or infinite) in the general case. The (negative) answer to this question is here developed in various ways. The main conclusions may be summarised as follows. When $q>1$, a_n can have lower limit $-\infty$ together with limit points everywhere dense in any given (finite or infinite) interval. If, however, a_n is bounded below, it is also bounded above and the s subsequences $\{a_{n_k}\}$ ($k=1, \dots, s$), where $s=(p, q)$, are convergent. The main argument consists in denoting the difference of the two sides of (1) by $e_n/pq \geq 0$ and expressing $\{a_n\}$ in terms of $\{e_n\}$ by solving the difference equation; but a more elementary alternative (on the lines of Leja's treatment of the case $q=1$) is given in the special case $p=3, q=2$. [The paper contains numerous mistakes, but these seem to affect irrelevant details rather than essentials and can be corrected or replaced by alternative arguments leading to the same conclusions.]

A. E. Ingham.

Tsuchikura, Tamotsu. *On asymptotically absolute convergence*. Tôhoku Math. J. (2) 3, 203-207 (1951).

The author introduces the following terminology. A series $\sum a_n$ of real numbers is said to be asymptotically absolutely convergent if there exists an increasing sequence $\{n_k\}$ with $n_k/k \rightarrow 1$ as $k \rightarrow \infty$ such that $\sum |a_{n_k}| < \infty$. The author proves that under certain conditions for the coefficients (e.g. $|a_n|$ monotonic) the asymptotic absolute convergence implies the absolute convergence. He also proves certain results for trigonometric series, such as: if $\sum A_n \sin nx$ converges absolutely at the point x_0 , with x_0/π irrational, then $\sum A_n$ is asymptotically absolutely convergent.

R. Salem.

Delange, Hubert, et Zamansky, Marc. *Sur une classe de procédés de sommation des séries divergentes*. C. R. Acad. Sci. Paris 234, 1025-1027 (1952).

The purpose of the note is to correct and complete a previous note by the second author [same C. R. 233, 999-1001 (1951); these Rev. 13, 455]. The authors consider the method of summability of series $\sum u_k$, which is defined by the expressions $T_n(g) = \sum_{k \leq n} g(k/x) u_k$. The function $g(u)$ here is defined for $0 \leq u \leq 1$ and x is a continuous parameter tending to $+\infty$. For $g(u) = (1-u)^p$, we get M. Riesz' definition of the method (C, p) . The authors state several results, all without proof. We shall quote the following one, in which $M(g, s) = \int_0^1 t^{s-1} g(t) dt$ is the Mellin transform of the function

equal to $g(t)$ in $(0, 1)$ and equal to 0 for $t > 1$. (i) Suppose that $g(u)$ and $G(u)$ are continuous on $(0, 1)$, and that the ratio $M(G, s)/M(g, s)$ is representable in the half-plane $\Re s > 0$ by a Laplace transform $\int_0^\infty e^{-st} d\alpha(x)$ which converges absolutely for $\Re s \geq 0$. Then summability (g) implies that of (G) . (ii) If the ratio $N(G, s)/M(g, s)$ is representable in strip $0 < \Re s \leq \beta + 1$ by a Laplace integral $\int_0^\infty e^{-st} d\alpha(x)$, which converges absolutely for $0 \leq \Re s \leq \beta + 1$, then every series $\sum u_k$ which is summable (g) and satisfies $u_k = o(k^\beta)$, is also summable (G) .
A. Zygmund (Chicago, Ill.).

Kuttner, B. On the "second theorem of consistency" for Riesz summability. II. J. London Math. Soc. 27, 207-217 (1952).

This is a sequel to the author's paper [same J. 26, 104-111 (1951); these Rev. 12, 696] where it was assumed that k is a positive integer. The problem is to characterize the asymptotic behavior of those locally decent functions $\varphi(t)$ for which the following proposition $P(k, \varphi)$ is true: If a series is evaluable by the Riesz method (R, λ_n, k) , then it is also evaluable by the Riesz method $(R, \varphi(\lambda_n), k)$. A familiar example is cited to show that the characterization which is valid when k is a positive integer becomes invalid when k is nonintegral. Assuming now that k is positive (integral or nonintegral), and assuming further local conditions on $\varphi(t)$, the author obtains a solution of the problem.
R. P. Agnew (Ithaca, N. Y.).

Aucoin, A. A. A generalization of Abel's transformation. Proc. Amer. Math. Soc. 3, 120-125 (1952).

The generalization is

$$\sum_{j=0}^n a_j b_j = \sum_{j=0}^{n-1} A_j (b_j - b_{j+1}) + A_{n-1} b_n - \sum_{i=0}^{p-1} \binom{p}{i+1} \sum_{j=0}^{n-1} A_j b_{j+i+1},$$

where $A_n = \sum_{j=0}^n a_j$ and p is a positive integer. Theorems on convergence of series, known for $p=1$, are deduced.

G. G. Lorents (Toronto).

Agnew, R. P. Abel transforms of Tauberian series and analytic approximation to curves and functions. Duke Math. J. 19, 131-138 (1952).

Let C denote a plane, closed (not necessarily simple) curve $z = z(t) = x(t) + iy(t)$, where $x(t)$, $y(t)$ are continuous, periodic functions, of bounded variation over a period. Let h be a positive number. Then there exists a closed rectifiable (in fact, analytic) curve $C_h: z = z_h(t)$ such that if the series $\sum_{n=0}^\infty u_n$ has the properties that $nu_n = h + o(1)$, as $n \rightarrow \infty$, and that $s_n = u_0 + \dots + u_n$ is on C for every n and progresses along C in the positive direction with increasing n , then the set of limit points of $\sigma(r) = (1-r) \sum_{n=0}^\infty s_n r^n$, as $r \rightarrow 1$, is the curve C_h . Relations between C and C_h , as $h \rightarrow 0$ or $h \rightarrow \infty$, are considered.
P. Hartman (Baltimore, Md.).

Pennington, W. B. A Tauberian theorem on the oscillation of Riesz means. J. London Math. Soc. 27, 199-206 (1952).

The following Tauberian theorem is proven. Let $\phi(x)$ be positive and increasing and let

$$\lim_{\rho \rightarrow \infty} \limsup_{\substack{x \rightarrow \infty \\ x \leq \rho}} x^{-\alpha} \phi(\rho, x) \frac{\Gamma(\alpha + \rho + 2)}{\Gamma(\alpha + 1) \Gamma(\rho + 1)} = A$$

for the function $\Phi(\rho, x) = \int_0^\infty \phi(u) (1 - u/x)^\rho du$, then

$$\phi(x) \sim A \Gamma(\alpha + 1) \Gamma(\rho + 1) x^\alpha / \Gamma(\alpha + \rho + 2) \text{ as } x \rightarrow \infty.$$

As a corollary, let $L^{(\alpha)}$ and $I^{(\alpha)}$ denote the upper and the lower limits of the Riesz $R(\lambda_n, \kappa)$ means of a series. If $L^{(\alpha)} \rightarrow A$, $I^{(\alpha)} \rightarrow A$, $\kappa \rightarrow \infty$, then for some κ_0 , $I^{(\alpha)} > -\infty$ and the series is $R(\lambda_n, \kappa)$ summable for $\kappa \geq \kappa_0 + 1$. [For the Cesàro means C_κ this result was known even with $\kappa > \kappa_0$, Littlewood, J. London Math. Soc. 10, 309-310 (1935)].

G. G. Lorents (Kingston, Ont.).

*Keldyš, M. V. On a Tauberian theorem. Trudy Mat. Inst. Steklov., v. 38, pp. 77-86. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

The following result is given. If $\lim_{x \rightarrow \infty} [f(x)/g(x)] = 1$ with $f(x) = \int_0^\infty (t-x)^{-\alpha} d\phi(t)$, $g(x) = \int_0^\infty (t-x)^{-\beta} d\psi(t)$, then $\lim_{x \rightarrow \infty} [\phi(x)/\psi(x)] = 1$. It is assumed that $\phi(x)$, $\psi(x)$ are positive and increasing, that $\alpha\phi(x) < x\phi'(x) < \beta\phi(x)$, $0 < \beta < \alpha + 1$, $m = [\beta]$. The proof depends on the properties of the integrals of the type $\int_0^\infty (t-x)^{-1-\alpha} \phi(t) dt$ in the complex plane. The author has applied this theorem for an estimation of eigenvalues of certain functional equations [Doklady Akad. Nauk SSSR (N.S.) 77, 11-14 (1951); these Rev. 12, 835].
G. G. Lorents (Kingston, Ont.).

Austin, M. C. On the absolute summability of a Dirichlet series. J. London Math. Soc. 27, 189-198 (1952).

Suppose that $\sum a_n n^{-s}$ is a Dirichlet series with

$$D = \limsup (\log n / \log l_n) < \infty.$$

Let $k \geq 0$ and let σ_k , δ_k be respectively the abscissas of summability (R, l, k) and $|R, l, k|$. Let k denote the lower bound of the numbers k such that $\sigma_k < +\infty$. The author proves that $\sigma_k \leq \sigma_k + D$ for all $k \neq k$ (for integral values of k proved by L. S. Bosanquet [same J. 22, 190-195 (1948); these Rev. 9, 581]). If $k < \alpha < \beta < \gamma$ one has $(\gamma - \alpha)\sigma_\beta \leq (\gamma - \beta)\sigma_\alpha + (\beta - \alpha)\sigma_\gamma$ so that either $\sigma_\beta = -\infty$ or σ_β is continuous and convex for $k > k$. If $0 < \delta \leq 1$ and $k \geq 0$, then $\sigma_{k+\delta} \leq \sigma_k + (1 - \delta)D$. A more general result is that if $\int_1^\infty (x-u)^k dA(u) = O(x^{k+\rho})$, then the series is summable $|R, l, k + \delta|$ for every $\sigma > p + (1 - \delta)D$. Here $A(u) = \sum a_n$ for $l_n < u$, $k \geq 0$, $k + \rho > 0$, and $0 < \delta \leq 1$.
E. Hille (New Haven, Conn.).

Macintyre, A. J., and Macintyre, Sheila Scott. Theorems on the convergence and asymptotic validity of Abel's series. Proc. Roy. Soc. Edinburgh. Sect. A. 63, 222-231 (1952).

If $F(x) \in C^\infty$, the series $(*) \sum_{n=0}^\infty F^{(n)}(n) z (z-n)^{-1/n} / n!$ is called the Abel series associated with $F(x)$. If F is entire and of sufficiently slow growth, $(*)$ converges or is summable to $F(z)$ [Buck, Trans. Amer. Math. Soc. 64, 283-298 (1948); these Rev. 10, 693]. In the general case, when the series $(*)$ converges its sum is an entire function $f(z)$ of exponential type whose growth function obeys $h(\theta, f) \leq b(\theta)$, where $b(\theta)$ is the supporting function of a certain convex set. The present paper discusses the convergence of $(*)$ and the relation of $F(x)$ and $f(z)$ when F is analytic in a (large) angle. Let $F(z)$ be analytic for $|\theta| \leq 3\pi/4$ with

$$|F(re^{i\theta})| \leq Kr^{-\gamma} \exp(rb(\theta)),$$

for some K , and $\gamma > 0$. The series $(*)$ is then convergent for all z . If, further, $F(z)$ is analytic for $x \geq -h$ and $\gamma > 1$, then $|F(z) - f(z)| \leq A/(x \log x)^h$ as $x \rightarrow +\infty$, so that $(*)$ represents F in an asymptotic sense. In each of these statements, the angular region in which F is required to be

analytic may be decreased, at the expense of more stringent growth restrictions on F . R. C. Buck (Madison, Wis.).

Chow, Hung Ching. A note on the summability of a power series on its circle of convergence. J. London Math. Soc. 26, 290-294 (1951).

$f(z) = \sum_{n=0}^{\infty} c_n z^n$ converging for $|z| < 1$ is said to belong to $\text{Lip}(k, p)$, $p > 0$, $0 < k \leq 1$, if $\int_0^{2\pi} |f(re^{i\theta})| r d\theta = O((1-r)^{-p+k})$ as $r \uparrow 1$. The author proves that if $f(z) \in \text{Lip}(k, p)$ with $k < 1$ then, for almost all θ , the series $\sum c_n e^{in\theta}$ is (1) Cesàro summable of order α for $\alpha > -k + \max(0, p^{-1} - 1)$; (2) Cesàro absolutely summable of order α for $\alpha > -k + \max(2^{-1}, p^{-1})$. These results, in particular (2), extend and complete earlier work by the author [J. London Math. Soc. 17, 17-23 (1942); these Rev. 4, 38]. A. Dvoretzky (Jerusalem).

Tomitch, Boško. Développement d'une puissance entière positive du monome en polynome des coefficients du binome. Bull. Soc. Math. Phys. Serbie 3, no. 1-2, 39-45 (1951). (Serbo-Croatian. French summary)
Starting with the formulas

$$\frac{1}{1-xe^a} = \sum_{n=0}^{\infty} \frac{a^n}{n!} \phi_n(x),$$

$$\frac{\psi_n(x)}{(1-x)^{n+1}} = \phi_n(x) = \sum_{k=0}^n k^n x^k,$$

$$\psi_n(x) = A_0^{(n)} + A_1^{(n)}x + \dots + A_n^{(n)}x^n,$$

many identities involving the constants $A_k^{(n)}$ are obtained by manipulations with power series and finite sums. The final results are

$$x^n = \sum_{k=0}^n A_{k+1}^{(n)} \binom{x+k}{n}, \quad B_n(x) = \sum_{k=0}^n A_{k+1}^{(n)} \binom{x+k}{n+1},$$

$$\sum_{k=1}^n k^n = \sum_{j=0}^n A_{j+1}^{(n)} \binom{p+1+j}{n+1},$$

the functions $B_n(x)$ being the Bernoulli polynomials.

R. P. Agnew (Ithaca, N. Y.).

Fourier Series and Generalizations, Integral Transforms

Sunouchi, Gen-ichirō. Notes on Fourier analysis. XLVI. A convergence criterion for Fourier series. Tôhoku Math. J. (2) 3, 216-219 (1951).

The author proves that if $\Delta \geq 1$ and

$$(*) \quad \int_0^t \varphi(u) du = o(t^\Delta) \text{ as } t \rightarrow +0,$$

$$(**) \quad \int_0^1 |d[u^\Delta \varphi(u)]| = O(t) \text{ in } (0, \eta),$$

then the Fourier series of the even function $\varphi(t)$ converges for $t=0$. The case $\Delta=1$ was given by Pollard, in extension of the well known criterion of W. H. Young [Pollard, J. London Math. Soc. 2, 255-262 (1927).] It is known that the result for $\Delta > 1$ would be false if $o(t^\Delta)$ in (*) were replaced by $o(t/\log t^{-1})$ [Randels, Ann. of Math. (2) 36, 838-858 (1935)]. L. S. Bosanquet (London).

Izumi, Shin-ichi. Notes on Fourier analysis. XXVII. A theorem on Cesàro summation. Tôhoku Math. J. (2) 3, 212-215 (1951).

It is known that: (1) if $\varphi(t) = o(t^\beta)$ (C, β), $\beta \geq 0$, $0 \leq \delta < 1$, then the Fourier series of the even function $\varphi(t)$ at $t=0$ satisfies $s_n = o(n^{-\delta})$ (C, α), $\alpha > \beta + \delta$; (2) if $s_n = o(n^{-\delta})$ (C, β), $\beta \geq 0$, $0 \leq \delta < 1$, $\beta - \delta > 0$, then $\varphi(t) = o(t^\alpha)$ (C, α), $\alpha > 1 + \beta - \delta$. [(1) was given by Obrechhoff, Bull. Soc. Math. France 62, 84-109 (1934); (2) was given by Hyslop, J. London Math. Soc. 24, 91-100 (1949); these Rev. 11, 100.] Here the author extends (2) to the range $\beta > -1$, $\delta > -1$, $\beta - \delta > -1$.

L. S. Bosanquet (London).

Tsuchikura, Tamotsu. Notes on Fourier analysis. XL. Remark on the Rademacher system. Proc. Japan Acad. 27, 141-145 (1951).

Let $\{r_n(x)\}$ be the system of Rademacher functions. Let $\{p_n\}$ be an increasing sequence of positive numbers, and let $P_n = p_1 + p_2 + \dots + p_n$. The main result of the note is that the functions

$$\varphi_n(x) = \{p_1 r_1(x) + p_2 r_2(x) + \dots + p_n r_n(x)\} / P_n$$

tend to zero at almost every point x , provided

$$(*) \quad p_n / P_n = o(1/\log \log P_n).$$

The author states that it had been proved by Maruyama [in a paper in Japanese] that the conclusion fails if the "o" in (*) is replaced by "O". A. Zygmund.

Levitan, B. M. A generalization of almost periodic functions. Amer. Math. Soc. Translation no. 63, 31 pp. (1952).

Translated from Mat. Sbornik N.S. 24(66), 321-346 (1949); these Rev. 11, 174.

Delange, Hubert. Sur un théorème de Widder. Bull. Sci. Math. (2) 76, 10-17 (1952).

Let $A(t)$ be a normalized nondecreasing function such that $F(x) = \int_0^\infty e^{-xt} dA(t)$ converges for $x > a$. Then a necessary and sufficient condition that $f(x) = \int_0^\infty e^{-xt} d\alpha(t)$, $x > a$, with $\alpha(t)$ of bounded variation on every finite interval, and $|\alpha(t'') - \alpha(t')| \leq M[A(t'') - A(t')]$, $0 \leq t' < t''$, is that $|f^{(n)}(x)| \leq M|F^{(n)}(x)|$. The case $F(x) = 1/x$ is the theorem of Widder. [Similar theorems and proofs are indicated by Boas [C. R. Acad. Sci. Paris 224, 1683-1685 (1947); these Rev. 8, 569] and Loève [ibid. 225, 31-33 (1947); these Rev. 9, 82]; these notes originated in a result of Tagamlitzki [ibid. 223, 940-942 (1946); these Rev. 8, 259] which is rediscovered by the author as the special case $f(x) = e^{-x}$.]

R. P. Boas, Jr. (Evanston, Ill.).

Zeller, Karl. Über Stetigkeit von Integraltransformationen. Math. Z. 55, 167-182 (1952).

The author is concerned with the continuity of the following integral transformation:

$$y(s) = \gamma(s) \cdot x(s) + \lim_{T \rightarrow \infty} \int_0^T \Gamma(s, t) x(t) dt$$

for $s \in [0, \infty)$. Here x, γ, Γ are complex-valued measurable functions and the integral over the finite range is taken to be a Lebesgue integral. Suppose now that the transformation is defined on a class of functions X which constitute a linear metric complete space (that is an F -space) and that the image functions are contained in some F -space Y . Then it is shown that the transformation is continuous on X to Y . An additional hypothesis is needed to insure that converg-

ence in X and Y at least implies a "natural" convergence. The theorem is given a rather ingenious proof which relies very heavily on the completeness assumption for F -spaces. The author also extends this theorem to more general types of spaces. *R. S. Phillips* (Los Angeles, Calif.).

Delerue, P. Calcul symbolique à 2 ou n variables et équations intégrales. *Ann. Soc. Sci. Bruxelles. Sér. I.* 65, 96-102 (1951).

The author expresses the (double) Laplace transforms of $x^{a-1}F(x^{-1}, y)$ and $x^{a-1}y^{b-1}F(x^{-1}, y^{-1})$ in terms of the (double) Laplace transform of $F(x, y)$ and utilizes his formulas for the solution of certain integral equations. *A. Erdélyi*.

Smith, J. J., and Alger, P. L. The use of the null-unit function in generalized integration. *J. Franklin Inst.* 253, 235-250 (1952).

Les auteurs rappellent diverses définitions de la dérivée d'ordre (≥ 0 ou < 0) non entier, et la nécessité d'un choix correct d'une origine et d'une dérivation au sens de la théorie des distributions, pour que ces dérivations commutent. *L. Schwartz* (Rio de Janeiro).

Mikusinski, J. G.-. Sur les fonctions exponentielles du calcul opératoire. *Studia Math.* 12, 208-224 (1951).

Dans le corps des fractions Q de l'algèbre de convolution des fonctions numériques continues sur $(0, +\infty)$ [*Studia Math.* 11, 41-70 (1949); *ces Rev.* 12, 189], un élément a est dit logarithme si l'équation différentielle $x'(\lambda) - ax(\lambda) = 0$, $x(0) = \text{unité}$, pour la fonction $x(\lambda)$ de la variable réelle λ à valeurs dans Q , a une solution; cette solution est alors unique et $x(1)$ est noté $\exp a$. L'article a pour but de démontrer que, si s (usuellement p) est l'opérateur de dérivation d/dx , considéré comme élément de Q , alors λs^a est un logarithme pour $a < 1$, ou pour $a = 1$ et λ réel, et ne l'est pas dans les autres cas. Ce théorème a des conséquences importantes dans la théorie des équations aux dérivées partielles ou intégral-différentielles. *L. Schwartz*.

Mikusinski, J. G.-., et Ryll-Nardzewski, C. Sur l'opérateur de translation. *Studia Math.* 12, 205-207 (1951).

Dans le corps Q , soit s l'élément d/dx [voir l'analyse ci-dessus]. Alors $\exp \lambda s$, défini par une équation différentielle, ne saurait être défini par la série $\sum_{k=0}^{\infty} (\lambda s)^k / (k!)$, divergente; mais on a $\exp(-\lambda s) = \lim_{n \rightarrow \infty} (1 + \lambda s/n)^{-n}$, si $\lambda \geq 0$.

L. Schwartz (Rio de Janeiro).

Polynomials, Polynomial Approximations

O'Donnell, Ruth E. A note on the location of the zeros of polynomials. *Proc. Amer. Math. Soc.* 3, 116-119 (1952).

Let the ratios $r_j = (-1)^j a_j / a_{j+1}$ be formed for all pairs a_j and a_{j+1} of successive non-zero coefficients of the polynomial $p(z) = \sum_{j=0}^n a_j z^j$. If all the r_j lie in the sector $S: |\arg z - \theta| \leq \pi/h$, then, as the author shows, all the zeros of $p(z)$ lie in the sector $S_0: |\arg z - \theta| \leq \pi - (\pi/n) + (\pi/h)$, and this is the best possible bound. The theorem is proved by showing that, if a zero z_0 of $p(z)$ were to lie outside S_0 , then the arguments of all terms of a certain sum would be numerically less or equal to $\pi/2$ with at least one actually less than $\pi/2$. This would imply the contradiction $p(z_0) \neq 0$. Similar results for the zeros of the k th derivative of $p(z)$ and for the k -fold zeros of $p(z)$ are obtained as corollaries of the theorem. *M. Marden* (Milwaukee, Wis.).

Sz.-Nagy, Gyula. Über Polynome mit lauter reellen Nullstellen. *Acta Math. Acad. Sci. Hungar.* 1, 225-228 (1950). (German. Russian summary)

For an n th degree polynomial $f(x)$ with only real zeros x_j , at least two of which are distinct, let D_n denote the maximum distance between the x_j , d_n the average distance between neighboring x_j , δ_n the mean square of these distances and ρ_n the dispersion of the x_j from their average A . The formulas defining these quantities are

$$d_n = (n-1)D_n, \quad N(n-1)\delta_n^2 = 2\sum (x_i - x_j)^2$$

and

$$n\rho_n^2 = \sum (x_j - A)^2.$$

Let D_{n-1} , d_{n-1} , δ_{n-1} and ρ_{n-1} denote the corresponding quantities for the k th derivative of $f(x)$. Then the author shows that $d_k < d_{k-1}$, $\delta_k > \delta_{k-1}$ and $\rho_k > \rho_{k-1}$ for $k = 3, 4, \dots, n$. The proof is based partly on the earlier result

$$D_{n-1}/D_n \geq [k(k-1)/n(n-1)]^{1/2}$$

of the author [*Jber. Deutsch. Math. Verein.* 27, 37-43 (1918)]. *M. Marden* (Milwaukee, Wis.).

Sz.-Nagy, Gyula. Realitätsgrad und Realitätsstellen von komplexen Polynomen. *Acta Math. Acad. Sci. Hungar.* 2, 99-103 (1951). (German. Russian summary)

Let an n th degree complex polynomial $f(z)$ have (counting multiplicities) m real zeros, p zeros with positive imaginary parts and q zeros with negative imaginary parts. By the "reality degree" of $f(z)$ the author means the number $r = m + |p - q|$ and by the "reality positions" the R points on the real axis where $f(z)$ takes on a real value. He shows that $R \geq r$, with the equality holding for at least one polynomial for which $n - r$ is even. If $q = 0$ and if u and v are two "reality positions," at least one non-real zero of $f(z)$ lies in the lens-region comprised of all points at which the segment connecting u and v subtends an angle of at least (π/p) . An analogous theorem is also proved for rational functions of which all the zeros lie above and all the poles below the real axis. The proofs are based on the variation in argument of the factors of the polynomial. *M. Marden*.

Sz.-Nagy, Gyula. Winkelabweichung und Betragsabweichung bei Polynomen. *Acta Math. Acad. Sci. Hungar.* 2, 11-18 (1951). (German. Russian summary)

Let $f(z)$ be an n th degree polynomial which is different from zero in a region R and let c be a fixed point in R . Let $W(c, z) = |\arg [f(z)/f(c)]|$, $B(c, z) = |f(z)/f(c)|$. The author defines the "angle variation" $A(c, R)$ and the "modulus variation" $M(c, R)$ of $f(z)$ in R relative to c as $\max W(c, z)$ and $\max B(c, z)$, z in R , respectively. If $f(z)$ has no zeros in the circle $|z - c| \leq r$, and R is the interior of the concentric circle of radius $r \sin(\omega/n)$, where $0 < \omega \leq \pi$, he shows that $A(c, R) < \omega$; whereas if R is the interior of the concentric circle of radius rt where $0 < t < 1$, he shows that $(1-t)^n \leq M(c, R) \leq (1+t)^n$. He derives also somewhat more involved inequalities for $A(c, R)$ and $M(c, R)$ based upon the positions of the individual zeros of $f(z)$. All the proofs are elementary. *M. Marden* (Milwaukee, Wis.).

Geronimus, Ya. L. On the orthogonal polynomials of V. A. Steklov. *Doklady Akad. Nauk SSSR (N.S.)* 83, 5-8 (1952). (Russian)

The author calls polynomials of Steklov those orthonormal polynomials $\{\varphi_n(x)\}$, $n = 0, 1, \dots$, relative to a weight function $w(x)$ on $[-1, 1]$, having the property that the set $\{\varphi_n\}$ is uniformly bounded on some (closed) interval

$[a, b]$, $-1 \leq a \leq x \leq b \leq 1$. (Steklov showed [Izvestiya Rossiiskoi Akad. Nauk (6) 15, 281-302 (1921)] that for such a system $\{\varphi_n\}$, if $x \in [a, b]$ and $F(x)$ satisfies a Lipschitz condition of order α ($0 < \alpha \leq 1$), then the Fourier-Tchebycheff series for $F(x)$ in terms of $\{\varphi_n\}$ converges to $F(x)$ at the point x .) The present work contains a sufficient condition on $w(x)$ in order that $\{\varphi_n\}$ be a Steklov set for $[a, b] = [-1, 1]$. Let $f(\theta)$ be a 2π -periodic function in the space $L_2(0, 2\pi)$, with usual norm $\|f\| = \{(1/2\pi) \int_0^{2\pi} |f(\theta)|^2 d\theta\}^{1/2} < \infty$, and set $\omega_2(f; \delta) = \sup_{0 < h \leq \delta} \|f(\theta+h) - f(\theta)\|$. Function f is said to belong to class $\text{Lip}(\alpha, 2)$ if $\omega_2(f; \delta) \leq M\delta^\alpha$, $0 < \alpha \leq 1$.

Let $p(\theta)$ be a weight function on the unit circle $|z| = 1$, and $\{P_n(z)\}$, $n = 0, 1, \dots$, the corresponding orthonormal system of polynomials:

$$\frac{1}{2\pi} \int_0^{2\pi} p(\theta) P_n(e^{i\theta}) \overline{P_m(e^{i\theta})} d\theta = \delta_{nm}.$$

A first theorem is that if $p(\theta) \in \text{Lip}(\alpha, 2)$ with $\frac{1}{2} \leq \alpha \leq 1$ and if $0 < m_1 \leq p(\theta) \leq m_2$ almost everywhere on $[0, 2\pi]$ then set $\{P_n(z)\}$ is uniformly bounded in $|z| \leq 1$. Now suppose $w(x)$ is a weight function on $[-1, 1]$ and define

$$(a) \quad p(\theta) = w(\cos \theta) \cdot |\sin \theta|,$$

so that the orthonormal set $\{\varphi_n\}$ corresponding to $w(x)$ is related to $\{P_n(z)\}$ by

$$\varphi_n(x) = \left\{ 2\pi \left[1 + \frac{P_{2n}(0)}{P_{2n}^*(0)} \right] \right\}^{-1/2} \frac{P_{2n}^*(z) + P_{2n}(z)}{z^n},$$

with $z = x + \sqrt{(x^2 - 1)}$ and $P_n^*(z) = z^n P_n(1/z)$. Theorem 2 then follows: $\{\varphi_n\}$ is a Steklov set if $p(\theta)$, given by (a), satisfies the conditions of Theorem 1. I. M. Sheffer.

*Vitali, G., e Sansone, G. *Moderna teoria delle funzioni di variabile reale. Parte II. Sviluppi in serie di funzioni ortogonali*. 3rd ed. Nicola Zanichelli Editore, Bologna, 1952. viii+614 pp. 7000 lire.

For a review of the 2d edition (1946) of this part, which is written by Sansone, see these Rev. 7, 434. In this edition several sections have been rewritten to take account of recent work and several new sections have been added.

Dinghas, Alexander. *Über eine Integralgleichung für die Polynome der Potentialtheorie*. Norske Vid. Selsk. Skr., Trondheim 1950, no. 2, 14 pp. (1951).

The Gegenbauer polynomials, which occur in potential theory, are defined by the generating function

$$(1 - 2hx + h^2)^{-\alpha} = \sum_{n=0}^{\infty} C_n^\alpha(x) h^n,$$

where the parameter α is positive. It is shown that, if

$$K(\theta, x) = \int_0^\pi \sin^{2\alpha-1} \psi \operatorname{cosec}^{2\alpha} \frac{1}{2} \rho d\psi,$$

where $\cos \rho = \cos x \cos \theta + \sin x \sin \theta \cos \psi$, then the polynomials $C_n^\alpha(\cos \theta)$ ($n = 0, 1, 2, \dots$) form the complete system of eigenfunctions of the integral equation

$$\varphi(\theta) = \frac{n+\alpha}{2^{2\alpha}\pi} \int_0^\pi K(\theta, x) \varphi(x) \sin^{2\alpha} x dx.$$

The proof depends on earlier work by the same author [Math. Z. 53, 76-83 (1950); these Rev. 12, 177].

E. T. Copson (St. Andrews).

Special Functions

*Titchmarsh, E. C. *The Theory of the Riemann Zeta-Function*. Oxford, at the Clarendon Press, 1951. vi+346 pp.

This book is an enlargement and revision of the author's Cambridge tract of 1930. Some new material, based on recent research, is added although the author does not pretend to cover the field exhaustively. As in his previous book, the ζ -function alone is treated with only one incidental reference to the more general Dirichlet L -functions being made; and, although the prime number theorem is proved, the error term is not sharpened by using the finer results on the ζ -function here proved. Indeed, number theory, which gave rise to the study of $\zeta(s)$, is highly subordinated. Nevertheless, the book is well written and the author is to be thanked both for making this material so readily available and for including more than one proof of a number of results. A description of the contents follows.

Chapter I, "The function $\zeta(s)$ and the Dirichlet series related to it," deals with identities connecting $\zeta(s)$ with number-theoretic functions. Chapter II, "The analytic character of $\zeta(s)$, and the functional equation," gives seven proofs of the functional equation relating $\zeta(1-s)$ to $\zeta(s)$. Chapter III, "The theorem of Hadamard and de la Vallée Poussin, and its consequences," gives these mathematicians' proofs that $\zeta(1+it) \neq 0$ and a proof of the prime number theorem. In addition, general theorems relating the order of $\zeta(s)$ in a region near $\sigma = 1$ to the zeros of $\zeta(s)$ and the orders of $\zeta'(s)/\zeta(s)$, $1/\zeta(s)$ near $\sigma = 1$ are given. Finally, the series for $1/\zeta(s)$, $\zeta'(s)/\zeta(s)$, $\log \zeta(s)$ and the Euler product for $\zeta(s)$ are investigated on the line $\sigma = 1$.

Chapter IV, "Approximate formulae," is concerned with the van der Corput estimation of integrals of the form $\int_a^b G(x) e^{i\psi(x)} dx$ and sums of the form $\sum_{a < n \leq b} g(n) e^{2\pi i \psi(n)}$. Also, several approximate functional equations for $\zeta(s)$ are given with that for $\zeta^2(s)$ being mentioned but not proved. Chapter V, "The order of $\zeta(s)$ in the critical strip," uses the exponential sum methods of Weyl and of van der Corput to estimate $\zeta(s)$ on various vertical lines, and it is shown that the latter's estimates are sharper. In particular, several estimates of $\zeta(\frac{1}{2} + it)$ are given but not the best result that is known. Chapter VI, "Vinogradoff's methods," deals with the exponential sum $\sum e^{2\pi i \psi(n)}$ and derives the (essentially) best published results on the zero-free region of $\zeta(s)$, namely, $\sigma > 1 - A/(\log t \log \log t)^{3/4}$. This chapter has no counterpart in the earlier tract. Chapter VII, "Mean-value theorems," deals with estimates and asymptotic expressions for the closely related integrals $\int_0^T |\zeta(\sigma + it)|^{2k} dt$ and $\int_0^T |\zeta(\sigma + it)|^{2k} e^{-\delta t} dt$. In the case $\sigma = \frac{1}{2}$, a lower bound for the second of these integrals is given and asymptotic estimates are given for both integrals when $k = 1, 2$. In addition, certain convexity theorems are proved.

Chapter VIII, "Ω theorems," uses the Dirichlet and Kronecker Diophantine approximation theorems and is more extensive than the corresponding part of the Cambridge tract. Among other things the following Ω result is proved: for fixed α and σ satisfying $0 < \alpha < 1 - \sigma \leq \frac{1}{2}$, it is false that $\zeta(s) = o(\exp \log^\alpha t)$. In chapter IX, "The general distribution of the zeros," various estimates of the function $N(\sigma, T)$ and $N(T) = N(1, T)$ are derived. The familiar estimate for $N(T)$ as $aT \log T + bT + 7/8 + S(T) + O(1/T)$ is obtained as are estimates for $\int_0^T S(t) dt$. It is also proved that for arbitrary positive h and large T , $N(T+h) - N(T) > K(h) \log T$. Here, again, the author is content to state certain results without proof. Chapter X, "The zeros on the critical line," gives lower

bounds for the number $N_0(T)$ of such zeros with ordinate not exceeding T . Hardy's result that $N_0(T) \rightarrow \infty$ as $T \rightarrow \infty$ is proved, as is A. Selberg's result that $N_0(T) > cT \log T$ for some positive c . Finally, a function having a functional equation similar to that for $\zeta(s)$ and possessing a number of other properties analogous to those of $\zeta(s)$ is constructed; this function has no Euler product and the Riemann hypothesis for it is false. Chapter XI, "The general distribution of the values of $\zeta(s)$," deals with the values taken on by $\zeta'(s)/\zeta(s)$ and $\log \zeta(s)$ on lines $\sigma = \sigma_0$ and for s in certain regions. The following result, together with others proved in the book, shows the exceptional nature of the zeros: for fixed $a \neq 0$, $\alpha > \frac{1}{2}$ and $\beta < 1$, the number of points in the rectangle $\alpha < \sigma < \beta$, $0 < t < T$ at which $\zeta(s) = a$ is greater than \sqrt{T} for some positive f .

Chapter XII, "Divisor problems," is concerned with the error $\Delta_k(x)$ in the formula $\sum_{n \leq x} d_k(n) = x P_k(\log x) + \Delta_k(x)$ where $d_k(n)$ is the number of ways of expressing n as a product of k factors and $P_k(y)$ is a polynomial in y of degree $k-1$. Estimates for $\Delta_k(x)$ and $\int_0^x \Delta_k^2(y) dy$ are given. None of this material appeared in the earlier tract. Chapter XIII, "The Lindelöf hypothesis," is concerned with necessary or sufficient conditions for the validity of this hypothesis that $\zeta(\frac{1}{2} + it) = O(t^\epsilon)$ for each positive ϵ . One necessary and sufficient condition is that $N(\sigma, T+1) - N(\sigma, T) = o(\log T)$ for each $\sigma > \frac{1}{2}$; other conditions are connected with the behavior of $\Delta_k(x)$. Chapter XIV, "Consequences of the Riemann hypothesis," derives bounds for $\zeta'(s)/\zeta(s)$, $\log \zeta(s)$, $\zeta(1+it)$, $\zeta(\frac{1}{2} + it)$, $1/\zeta(1+it)$ and some Ω results under this hypothesis. Also, some results on $\sum_{n \leq x} \mu(n)$ are obtained both with and without this hypothesis. The concluding chapter, "Calculations relating to the zeros," provides a brief description of our numerical knowledge about the zeros. A fourteen page bibliography including references given in the Cambridge tract but not in Landau's Handbuch is appended. There is no index. The typography is excellent. *L. Schoenfeld.*

Turán, P. On Carlson's theorem in the theory of the zeta-function of Riemann. *Acta Math. Acad. Sci. Hungar.* 2, 39-73 (1951). (English. Russian summary)

Estimates of the number $N(\sigma_0, T)$ of zeros of the Riemann zeta-function in the domain $\sigma_0 \leq \sigma < 1$, $0 < t \leq T$, play an important part in questions of prime-number theory, and have been given by Carlson and several later writers. The result obtained in this paper is

$$N(\sigma, T) = o(T^{2(1-\sigma)} + 600(1-\sigma)^{101/100} \log^4 T),$$

valid if $1-B \leq \sigma \leq 1$, where B is sufficiently small. The proof, which is rather intricate, depends essentially on the lemma: for $m \geq n$, ν an integer, $1 = |z_1| \geq |z_2| \geq \dots \geq |z_n|$, we have

$$\max_{1 \leq \nu \leq m+n} |d_1 z_1^\nu + \dots + d_n z_n^\nu| \geq \left(\frac{n}{e^4 m}\right)^n \min_{j=1, \dots, n} |d_1 + \dots + d_j|.$$

Further results depending on the Lindelöf hypothesis are also stated.

E. C. Titchmarsh (Oxford).

Nanjundiah, T. S. Certain summations due to Ramanujan, and their generalisations. *Proc. Indian Acad. Sci., Sect. A.* 34, 215-228 (1951).

The author establishes certain identities which are generalizations of formulae due to Ramanujan, of which the follow-

ing is typical:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{2q-1}} \left[a^{2q-1} \operatorname{cosech} \frac{n\pi b}{a} + (-1)^n b^{2q-1} \operatorname{cosech} \frac{n\pi a}{b} \right] \\ = -\frac{2}{\pi ab} \sum_{k=0}^q (-1)^k (1-2^{1-2k}) \zeta(2k) (1-2^{1-2q+2k}) \zeta(2q-2k) a^{2q-2k} b^{2k},$$

where $a > 0$, $b > 0$, and $q = 1, 2, 3, \dots$, ζ being the Riemann ζ -function. The proofs make use of the partial fraction expansions of the hyperbolic functions plus elementary manipulations of series. In two places modular identities (ascribed by the author to Ramanujan) are used. These are actually the transformation equations ($\tau \rightarrow -1/\tau$) of the modular functions $\eta(\tau)$ and $\eta^2(2\tau)/\eta(\tau)\eta(4\tau)$, where η is the Dedekind function. *J. Lehner (Philadelphia, Pa.).*

Lakin, A. A hypergeometric identity related to Dougall's theorem. *J. London Math. Soc.* 27, 229-234 (1952).

A particular ${}_7F_6$ with unit argument is summed by the method used in a previous note with Burchinal [*Quart. J. Math., Oxford Ser. (2)* 1, 161-164 (1950), these *Rev.* 12, 178]. There is a corresponding identity for basic series, proved by using q -difference equations. These are used also to obtain a simple proof of the basic analogue of Saalschütz's theorem and several related identities.

N. J. Fine (Philadelphia, Pa.).

Jackson, Margaret. Transformations of series of the type ${}_2H_1$ with unit argument. *J. London Math. Soc.* 27, 116-123 (1952).

H is Bailey's notation [*Quart. J. Math., Oxford Ser. (1)* 7, 105-115 (1936)] for a series that is like a generalized hypergeometric series except that summation is over both positive and negative powers of z . The author establishes certain transformations of ${}_2H_1$ with unit argument which generalize some of the two- and three-term relations of series ${}_2F_2$ given by Thomae and systematized by Whipple. She concludes that each of the older results admits of several generalizations and thinks it unlikely that the numerous formulas can be tabulated in any systematic way.

A. Erdélyi (Pasadena, Calif.).

Jackson, Margaret. A note on the sum of a particular well-poised ${}_4H_3$ with argument -1 . *J. London Math. Soc.* 27, 124-126 (1952).

The author sums in closed form the bilateral general hypergeometric series of argument -1 :

$${}_4H_3 \left[\begin{matrix} 1+\frac{1}{2}a, \frac{1}{2}+a-s, s+x, s-x, s+y, s-y; -1 \\ \frac{1}{2}a, \frac{1}{2}+s, 1+a-s-x, 1+a-s+x, \\ 1+a-s-y, 1+a-s+y \end{matrix} \right]$$

and gives several special cases in which ${}_4H_3$ reduces to ${}_2H_2$ with $p=3, 4, 5$.

A. Erdélyi (Pasadena, Calif.).

Campbell, Robert. Sur un cas de confluence des fonctions de Mathieu associées. *C. R. Acad. Sci. Paris* 234, 695-697 (1952).

Since a paraboloid of revolution is the limit of spheroids, certain confluent hypergeometric functions are limits of associated Mathieu functions. This connection leads the author to an inequality for zeros of associated Mathieu functions and also to a (known) integral equation for Laguerre polynomials. *A. Erdélyi (Pasadena, Calif.).*

Harmonic Functions, Potential Theory

Myrberg, Lauri. Über die vermischte Randwertaufgabe der harmonischen Funktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 103, 8 pp. (1951).

Existence and uniqueness theorems are established for the following mixed boundary value problem: Let $\{E_n\}$ denote a set of disjoint open arcs of the unit circumference such that the complement of $\bigcup E_n$ with respect to the unit circumference has capacity zero. It is supposed that the family of arcs E_n is partitioned into two classes $\{A_n\}$ and $\{B_n\}$. Further, $A(s)$ is given as bounded on $\bigcup A_n$ and continuous on the union of a countable set of open disjoint subarcs of the A -arcs which cover $\bigcup A_n$ save for a set M_1 of capacity zero; $B(s)$ is given as bounded and integrable in the sense of Lebesgue on $\bigcup B_n$. Let M_2 denote the set of $s = e^{i\theta} \in \bigcup B_n$ of measure zero for which it is not the case that $\lim_{h \rightarrow 0} h^{-1} \int_0^{2\pi} B(e^{it}) dt = B(s)$. It is required to construct a harmonic function u in $|z| < 1$ satisfying

$$\lim_{r \rightarrow 1} \frac{\partial u}{\partial r} = A(s), \quad \text{se } \bigcup A_n - M_1; \quad \lim_{r \rightarrow 1} u = B(s), \quad \text{se } \bigcup B_n - M_2.$$

Here the first limit is understood to be in the ordinary sense, the second as sectorial. Generalizations are indicated.

M. Heins (Paris).

Myrberg, Lauri. Bemerkungen zur Theorie der harmonischen Funktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 107, 8 pp. (1952).

Theorems pertaining to symmetry properties of certain classes of harmonic functions as well as theorems pertaining to the maximum principle and related questions are established. The following are typical. (1) Let G denote a region of the extended plane whose frontier lies on the real axis and has zero Lebesgue linear measure. Then a bounded harmonic function in G attains equal values at points which are symmetric with respect to the real axis. If the linear measure of the frontier is positive, there exist bounded harmonic functions in G not possessing this symmetry property. (2) If u is harmonic in the interior of the unit circle and has a finite Dirichlet integral, then $\lim_{r \rightarrow 1} u(re^{i\theta}) \leq M$ almost everywhere implies $u \leq M$ in the interior of the unit circle. The proofs use known results from the theory of bounded analytic functions (Painlevé, Denjoy), the Poisson integral, Green's formula. Related results using the notions of span [Schiffer, Duke Math. J. 10, 209-216 (1943); these Rev. 4, 271] and capacity are established.

M. Heins (Paris).

Lohwater, A. J. A uniqueness theorem for a class of harmonic functions. Proc. Amer. Math. Soc. 3, 278-279 (1952).

Let $u(r, \theta)$ be harmonic in $r < 1$. The following theorem is proved. If (1) there exists a constant M such that $\int_0^{2\pi} |u(r, \theta)| d\theta < M$, (2) $\lim_{r \rightarrow 1} u(r, \theta) = 0$ p.p. on $[0, 2\pi]$, (3) $\limsup_{r \rightarrow 1} u(r, \theta) > -\infty$, $\liminf_{r \rightarrow 1} u(r, \theta) < +\infty$, except on a countable set $E = \bigcup \theta_n$, then there exist real numbers c_n such that $\sum |c_n| < \infty$ and $u(r, \theta) = \sum_{n=1}^{\infty} c_n K(r, \theta - \theta_n)$, where $K(r, \theta)$ is Poisson's kernel. The proof depends on the known fact that a harmonic function satisfying (1) can be represented by a Poisson-Stieltjes integral with respect to a function μ of bounded variation; (2) and (3) then imply that μ is a step function.

W. Rudin (Cambridge, Mass.).

Mihlin, S. G. On an inequality for the boundary values of harmonic functions. Uspehi Matem. Nauk (N.S.) 6, no. 6(46), 158-159 (1951). (Russian)

Let $u = u(x_1, \dots, x_n)$ be harmonic when $x_n > 0$, $O(|x|^{-\alpha})$ when $|x| \rightarrow \infty$ and sufficiently well-behaved when $x_n = 0$. Then for some constant C ,

$$\int_{x_n=0} (u_1^2 + \dots + u_{n-1}^2 - C u_n^2) dx_1 \dots dx_{n-1} \leq 0.$$

A reverse inequality for the unit sphere has been proved by Višik [same journal 6, no. 2(42), 165-166 (1951); these Rev. 13, 235].

L. Gårding (Lund).

Ugaheri, Tadashi. On the general potential and capacity. Jap. J. Math. 20, 37-43 (1950).

Let $\mu(s)$ denote a p. d. m. (positive distribution of mass) on a bounded Borel set E in Euclidean space Ω , and let $\phi(r)$ be a strictly increasing, positive and continuous function for $r > 0$, such that $\phi(0+) = +\infty$. Then $U_\mu(P) = \int_E \phi(PQ) d\mu(Q)$ is the (generalized, ϕ) potential of μ with respect to $\phi(r)$. The author shows that the ϕ -potential has some of the basic properties of the Newtonian potential; he obtains analogues of some results due to Frostman [Potentiels d'équilibre . . . , Lund, 1935]. For example, if $U_\mu(P)$ is continuous in the kernel of μ , then $U_\mu(P)$ is continuous in Ω . The author also defines the (generalized, ϕ) capacity of E as

$$C(E) = \inf [\sup U_\mu(P) | \mu(E) = 1],$$

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and shows it has the usual properties. For example, $C(E)$ is positive if and only if there exists a p. d. m. μ on E , with $\mu(E) > 0$, such that $U_\mu(P)$ is finite everywhere. The author concludes by defining the (generalized, ϕ) transfinite diameter in the usual way and shows it has the usual basic properties. He also obtains an analogue of Evans' classic result on sets of zero capacity and positive infinite singularities of harmonic functions [see Rudin, Proc. Amer. Math. Soc. 2, 967-970 (1951); these Rev. 13, 555]. The results in this paper should be compared with those in Deny, Acta Math. 82, 107-183 (1950); these Rev. 12, 98.

M. Reade.

Simola, Inkeri. Potentialtheoretische Randwertprobleme für mehrfach zusammenhängende Gebiete. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 99, 50 pp. (1951).

On sait que la théorie de la représentation conforme sur le cercle-unité d'un domaine de Jordan simplement connexe G permet d'énoncer des résultats précis sur les fonctions harmoniques dans G : Représentation du type de Poisson-Stieltjes, théorème de Fatou (les limites radiales étant remplacées par les limites le long des trajectoires orthogonales des lignes de niveau de la fonction de Green g de pôle $s_0 \in G$), principe de Picard-Bouligand, etc. Tout ceci peut s'étendre au cas où G n'est plus simplement connexe, mais est limité par un nombre fini de courbes simples de Jordan: c'est l'objet de cet article, qui précise en divers points les premiers chapitres du traité classique de Nevanlinna. Un lemme essentiel est que les lignes de niveau de h (où $-h$ est harmonique conjuguée de g) aboutissent en des points bien déterminés de la frontière (extension du résultat bien connu de Carathéodory). Les méthodes employées sont élémentaires: ainsi on commence par construire la mesure harmonique d'un arc-frontière à l'aide du procédé alterné de Schwarz; d'ailleurs, d'une façon générale, il s'agit surtout d'un exposé didactique de résultats plus ou moins connus. Il serait désirable d'envisager le cas général où G est un domaine quelconque de R^n ($n \geq 2$); cela exigerait une étude

préliminaire des trajectoires orthogonales des "sphères de Green", étude entreprise récemment par Brelot et Choquet [C. R. Acad. Sci. Paris 228, 1556-1557 (1949); ces Rev. 11, 107].
J. Deny (Strasbourg).

Müller, Claus. Die Potentiale einfacher und mehrfacher Flächenbelegungen. Math. Ann. 123, 235-262 (1951).

Let F be an oriented region of an analytic surface in space of 3 dimensions and let σ be a point function defined on F . Let r_{st} be the distance between the point (x) in space and the point (ξ) on F . Representing the normal derivative at (ξ) by $\partial/\partial n_\xi$, the author defines the potential of a K -fold distribution on F as

$$U_K(x) = \int_F \sigma(\xi) \left(\frac{\partial}{\partial n_\xi} \right)^{K-1} \frac{1}{r_{st}} dF_\xi.$$

Further restrictions on F and $\sigma(\xi)$ are later imposed. Special attention is given to the classical cases in which $K=1$ or $K=2$. In the development of the theory a cartesian coordinate system is used together with a related coordinate system, u^1, u^2, u^3 . Tensor methods are used extensively. The author investigates the jump in the value of U_{K+1} and its derivatives in passing through F . The author also calls particular attention to a number of integral relations for potential functions. F. W. Perkins (Hanover, N. H.).

Differential Equations

San Juan, Ricardo. Le problème de Watson pour les solutions des équations différentielles linéaires homogènes. C. R. Acad. Sci. Paris 234, 1338-1340 (1952).

It is known that a linear homogeneous differential equation of order m with polynomial coefficients has m solutions $y(x)$ such that $y(x)e^{-\lambda x} \sim \sum_{k=0}^{\infty} a_k x^{-k}$ along the real axis. The author states, with indications of proofs, a theorem on the validity of the asymptotic expansion in a right half-plane, together with some consequences.

R. P. Boas, Jr. (Evanston, Ill.).

Germay, R. H. Remarque sur les intégrales infiniment voisines des équations différentielles de forme normale dépendant d'un paramètre variable. Bull. Soc. Roy. Sci. Liège 20, 392-399 (1951).

Consider (A) $dx_i/dt = \varphi_i(t, x, \lambda)$ where the φ_i are expandable as power series in $x = (x_1, \dots, x_n)$ and λ , with coefficients continuous in t , and $\varphi_i(t, 0, 0) = 0$. Let

$$x_i(t, \lambda) = \sum_{n=1}^{\infty} x_{i,n}(t) \lambda^n$$

be the solution of (A) which vanishes for $t=0$. The author studies explicit formulae for the $x_{i,n}$ in terms of the general solution of the system obtained when one replaces the right hand side of (A) by its linear part. F. M. Stewart.

Hukuhara, Masuo. Sur un système de deux équations différentielles ordinaires non linéaires à coefficients réels. J. Fac. Sci. Univ. Tokyo. Sect. I. 6, 295-317 (1951).

The author studies the system

$$(A) \quad \frac{dx}{dt} = \sum_{j+k \geq 1} p_{jk} x^j y^k, \quad \frac{dy}{dt} = \sum_{j+k \geq 1} q_{jk} x^j y^k,$$

where the p_{ij}, q_{ij} are real and the roots of

$$(p_{10} - \rho)(q_{01} - \rho) - p_{01}q_{10} = 0$$

are pure imaginary. Formal manipulation with power series is used to replace (A) by an equivalent equation,

$$(B) \quad \frac{dw}{dt} = w \left\{ i + \sum_{j=1}^{\infty} c_j (w\bar{w})^j \right\},$$

where $c_j = a_j + ib_j$ and w is a complex-valued function of the real variable t . Further reductions show that (B) is equivalent to one of the following normal forms: (a) $d\zeta/dt = i\zeta$, when $c_1 = c_2 = \dots = 0$; (b) $d\zeta/dt = i\zeta \{1 + b(\zeta\bar{\zeta})^n\}$, when

$$a_1 = a_2 = \dots = 0, \quad b_1 = \dots = b_{n-1} = 0, \quad b_n \neq 0;$$

$$(c) \quad d\zeta/dt = \zeta \left\{ i + i \sum_{j=1}^n b_j' (\zeta\bar{\zeta})^j + a(\zeta\bar{\zeta})^n + a'(\zeta\bar{\zeta})^{2n} \right\}, \quad a \neq 0$$

when $a_1 = \dots = a_{n-1} = 0, a_n \neq 0$. The origin is a center for (a) and (b) and a focus for (c). The normal form associated with a given equation is essentially unique. The last half of the paper is devoted to justifying the formal manipulations.

F. M. Stewart (Providence, R. I.).

Hukuhara, Masuo. Le problème aux limites d'un système de deux équations différentielles ordinaires. J. Math. Soc. Japan 3, 99-103 (1951).

A class of sets in 3-dimensional space is defined so that if E belongs to the class and f and g are continuous on E then the system $y' = f(x, y, z), z = g(x, y, z)$ has a solution with $y(a) = z(b) = 0$. Apparently the concept of "fonctions S positives de M. Kamke" is that introduced in the S. B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1930, no. 17 (1931).

F. M. Stewart (Providence, R. I.).

Saito, Toshiya. Differential equations with invariant Pfaffian forms. Kōdai Math. Sem. Rep. 1951, 103-117 (1951).

Let $x = (x_1, \dots, x_n)$ be the analytic local coordinates of a compact n -dimensional analytic manifold without torsion Ω ; let (1): $\dot{x} = X(x)$ be a system of differential equations, X one-valued real holomorphic; let S_i be the mapping $x = f(x^0, t)$, f being the general solution of (1). Suppose (1) admits $n-1$ linearly independent invariant exact Pfaffian forms $\omega_i = \sum A_{ik}(x) dx_k, A_{ik}$ one-valued real holomorphic,

$$\sum A_{ik} X_k = 0, \quad \partial A_{ik} / \partial x_j = \partial A_{ij} / \partial x_k.$$

Let Γ_k be a basis of the 1-Betti group, $\int_{\Gamma_k} \omega_i = \omega_{ik}$. Theorems 1 and 2: If there are real numbers λ_i (not all $=0$) such that either a) the numbers $\sum \lambda_i \omega_{ik}$ have rational ratios, or b) there are points x for which all $\sum \lambda_i A_{ik} = 0$, then S_i has an invariant closed analytic submanifold of dimension $\leq n-1$. Theorem 3: In order that any trajectory of (1) be dense in Ω it is necessary and sufficient that neither assumption a) nor b) be satisfied. Theorem 4: If Ω is an n -torus, S_i has a surface of section and assumptions a) and b) are not satisfied, then S_i is ergodic. Two particular applications of these theorems are worked out. J. L. Massera (Montevideo).

Sears, D. B. Some properties of a differential equation. J. London Math. Soc. 27, 180-188 (1952).

Wintner [J. London Math. Soc. 25, 347-351 (1950); these Rev. 12, 500] has shown that if $f(x)$ is continuous for $0 \leq x < \infty$ and $x^{3/2}f(x)$ is of class $L^1(0, \infty)$, then no solution ($\neq 0$) of (1) $y'' - f(x)y = 0$ is of class $L^2(0, \infty)$. The author generalizes this result as follows: if $p \geq 1, 1/p + 1/q = 1, n \geq 0$ and

$$|f(x)| \leq x^{-2} p^{-2} \left(p+1 + (p+2) \sum_{j=1}^n (\log x \log_1 x \cdots \log_j x)^{-1} \right) + \alpha(x)$$

for $x \geq x_0$, where $x\alpha(x)$ is of class $L^1(x_0, \infty)$ and $x^{2-1/p}\alpha(x)$ is of class $L^2(x_0, \infty)$, then (1) has no solution $y(x) \neq 0$ of class $L^p(0, \infty)$. (If $p=1$, then $q=\infty$ and L^∞ is the class of bounded functions.) The constants $(p+1)/p^2$ and $(p+2)/p^3$ cannot be improved. In fact, if $f(x) > 0$ and

$$f(x) \geq x^{-2}p^{-2} \left(a_0 + \sum_{j=1}^n a_j (\log x \log_2 x \cdots \log_j x)^{-1} \right) + \beta(x),$$

where $x\beta(x)$ is of class $L^1(x_0, \infty)$ and $a_0 > p+1$ when $n=0$ or $a=p+1$, $a_1=\cdots=a_{n-1}=p+2$, $a_n > p+2$ when $n>0$, then (1) possesses a solution $y=y(x) \neq 0$ of class $L^p(0, \infty)$. (These results are related to those of Hartman, Amer. J. Math. 73, 957-962 (1951); these Rev. 13, 463.) P. Hartman.

Manacorda, Tristano. Sul comportamento asintotico di una classe di equazioni differenziali lineari non omogenee. Boll. Un. Mat. Ital. (3) 6, 304-311 (1951).

Wintner [Amer. J. Math. 71, 595-602 (1949); these Rev. 11, 33] has shown that if $F(T) = \int_T^\infty f(t)dt$ has a finite limit $F(\infty)$, as $T \rightarrow \infty$, and if $\int^\infty \max_{t \geq t} |F(T) - F(\infty)| dt < \infty$, then $x'' + f(t)x = 0$ has a pair of solutions satisfying $x_1 = 1 + o(1)$, $x_1' = o(1)$ and $x_2 = t + o(1)$, $x_2' = 1 + o(1)$. By adapting Wintner's method of successive approximations, Manacorda shows that if, in addition, some solution $x = x_0(t)$ of $x'' = \varphi(t)$ has the property that $\Psi(T) = \int_T^\infty x_0(t)f(t)dt$ has a limit $\Psi(\infty)$ satisfying $\int^\infty \max_{t \geq t} |\Psi(T) - \Psi(\infty)| dt < \infty$, then there exists a particular solution of $x'' + f(t)x = \varphi(t)$ satisfying $x = x_0(t) + 1 + o(1)$, $x' = x_0'(t) + o(1)$. P. Hartman.

Leighton, Walter. On self-adjoint differential equations of second order. J. London Math. Soc. 27, 37-47 (1952).

The author's first theorem is that if $y(x)$, $z(x)$ are continuously differentiable functions for which the Wronskian, $y'z - yz'$, is not zero, then the zeros of y and z separate each other. (This does not depend on the author's assumption that y and z are solutions of linear, second order differential equations.) Several comparison theorems are deduced from this remark. It is well known that if $p > 0$ is monotone and $y(x) \neq 0$ is any solution of (1) $y'' + p(x)y = 0$, then $y^2 + y'^2/p$ and $1/p$ are monotone in the same sense. In particular, $y(x)$ remains bounded, as x increases, when p is non-decreasing. As a complementary result, Leighton shows that if p is non-increasing and y_1, y_2 are linearly independent solutions of (1), then $u = (y_1^2 + y_2^2)^{1/2}$ remains bounded from below, as x increases. These and other results are applied to (1) after standard changes of dependent and independent variables to obtain new theorems. P. Hartman.

Basov, V. P. Necessary and sufficient conditions for the stability of solutions of a certain class of systems of linear differential equations in one doubtful case. Doklady Akad. Nauk SSSR (N.S.) 81, 5-8 (1951). (Russian)

The following system of linear differential equations is considered (S): $dx_s/dt = \sum_{r=1}^n [p_{sr} + t^{-\gamma} q_{sr}(t)]x_r$, $s=1, \dots, n$, where γ, p_{sr} are real constants, $\gamma > 0$, and the q_{sr} are real continuous, bounded functions defined for $t \geq T > 0$. It is assumed that the characteristic equation for the matrix of constants (p_{sr}) has one zero root, with the remaining roots possessing negative real parts. Necessary and sufficient conditions are given for the stability and asymptotic stability (in the sense of Liapounov) of the zero solution $x_1 = \cdots = x_n = 0$ of (S). The proof follows from results obtained previously by the author [same Doklady 80, 301-304 (1951); these Rev. 13, 557] concerning the asymptotic nature of the solutions of (S). E. A. Coddington.

Simanov, S. N. On the theory of quasiharmonic oscillations. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 129-146 (1952). (Russian)

Let there be given a system

$$(1) \quad \dot{x}_s = \sum (a_{ss} + \mu p_{ss}(t, \mu))x_s + f_s(t, \mu), \quad s=1, 2, \dots, n$$

where: (a) the a_{ss} are constants; (b) the p_{ss}, f_s are continuous and of period 2π in t , and have Fourier expansions in t ; (c) the same variables are analytic functions of μ in $|\mu| \leq \mu^*$. The problem is to find solutions of (1) which are periodic and of period 2π in t . This question arises in certain physical and engineering applications [see notably: Timoshenko, Vibration problems in engineering, 2nd ed., Van Nostrand, New York, 1937; Kočin, same journal 2, 3-28 (1934)]. It arises also in connection with the search for non-linear oscillations by the perturbation method [see the monograph by Malkin, The methods of Lyapunov and Poincaré in the theory of non-linear oscillations (Russian), Moscow-Leningrad, 1949, §§5, 9, 10; these Rev. 12, 28].

In the present paper it is assumed that $\|a_{ss}\|$ has k characteristic roots zero with simple invariant factors and that among the remaining $l=n-k$ none are critical (i.e. zero or of the form $\pm Ni$). One may then, by a linear transformation of coordinates, replace (1) by an equivalent system

$$(2) \quad \begin{cases} \dot{u}_s = \mu(\sum p_{ss}u_s + \sum p_{s, k+j}v_j) + f_s(t, \mu), \\ \dot{v}_j = \sum c_{js}v_s + \mu(\sum p_{k+j, s}u_s + \sum p_{k+j, k+l}v_l) + f_{k+j}(t, \mu); \end{cases} \\ s=1, 2, \dots, k; \quad j=1, 2, \dots, l.$$

Here the c_{js} are constant and the characteristic roots of $\|c_{js}\|$ are all non-critical.

Side by side with (2) the author introduces the auxiliary system (3) in the z_s, y_j and k constants W_s of which the first k relations are

$$\dot{z}_s = \mu(\sum p_{ss}z_s + \sum p_{s, k+l}y_l) + f_s(t, \mu) + W_s,$$

and the last l are the same as in (2) with z, y replacing u, v . Let β_1, β_{k+l} be a set of initial values for the z_s, y_{k+l} . Theorem. Given β_1, \dots, β_k , one may select a unique set of values for the β_{k+l} and W_s such that the corresponding solution z_s^*, y_j^* of (3) be periodic and of period 2π . This solution will be linear (non-homogeneous) in β_1, \dots, β_k with coefficients analytic in μ in $|\mu| \leq \mu^*$, except for a finite set of points which do not include $\mu=0$. The appropriate W_s are given by

$$W_s = \frac{-1}{2\pi} \int_0^{2\pi} \{ \mu(\sum p_{ss}z_s^* + \sum p_{s, k+l}y_l^*) + f_s(t, \mu) \} dt.$$

To obtain z_s^*, y_j^* one writes down expansions in powers of μ :

$$z_s^* = z_s^0(t) + \mu z_s^1(t) + \dots, \quad y_j^* = y_j^0(t) + \mu y_j^1(t) + \dots, \\ W_s = W_s^0 + \mu W_s^1 + \dots,$$

and substitutes in (3). The resulting systems of differential equations are then solved one at a time for the z_s^r, y_j^r and the W_s^r selected each time to make the solutions periodic with period 2π . The W_s thus obtained assume the form

$$W_s = W_{s1}(\mu)\beta_1 + \dots + W_{sk}(\mu)\beta_k + W_{s0}(\mu),$$

where the $W_{sk}(\mu)$ are analytic in μ at $\mu=0$. If the relations $W_s=0$ in β_1, \dots, β_k are compatible they will yield the values of the β_k for which z_s^*, y_j^* become a periodic solution of (2).

S. Lefschetz (Princeton, N. J.).

Shimizu, Tatsujiro. On differential equations for non-linear oscillations. I. Math. Japonicae 2, 86-96 (1951).

The author states the following theorem: If the coefficient functions $f(x), g(x), h(t)$ of (1) $d^2x/dt^2 + f(x)dx/dt + g(x) = h(t)$

satisfy $f(x) \geq \text{const.} > 0$; $g(0) = 0$, $dg/dx > 0$ and $|g(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$; $h(t)$ is periodic of period p ; then (1) has at least one periodic solution of period p . (The proof, depending on geometrical arguments, is difficult to follow.) Variants of this theorem are obtained, for example, when $g(x)$ has an infinity of zeros with sufficiently large "oscillations" between zeros. The methods developed are then used to prove the existence of certain types of periodic solutions for $d^2x/dt^2 + dx/dt + cx + dx^3 = h \sin 2\pi t/p$, where suitable restrictions are imposed on ϵ, c, d, h . *P. Hartman.*

Fifer, Stanley. Studies in nonlinear vibration theory. J. Appl. Phys. **22**, 1421-1428 (1951).

The author studies the variational equation associated with the periodic solutions of

$$(A) \quad \ddot{x} + x = \epsilon \left\{ \left(\dot{x} - \frac{1}{2} \dot{x}^3 + \lambda \dot{x}^5 \right) + E \cos(\omega t + \psi) \right\}.$$

Perturbation methods are used to establish the existence of certain periodic solutions of the variational equation. Its almost periodic solutions are also investigated. There are diagrams showing the stability behavior of solutions of (A).

F. M. Stewart (Providence, R. I.).

Duff, G. F. D., and Levinson, N. On the non-uniqueness of periodic solutions for an asymmetric Liénard equation. Quart. Appl. Math. **10**, 86-88 (1952).

Let $g(x)$ be continuous for $-\infty < x < \infty$ and satisfy $xg(x) > 0$. Let $f(x)$ be continuous for $-\infty < x < \infty$ and subject to the inequalities $f(x) > 0$, $f(x) < 0$ or $f(x) > 0$ according as $x < -a$, $-a < x < b$ or $x > b$, where $a > 0$, $b > 0$. The authors give an example of such a pair of functions with the property that $d^2x/dt^2 + f(x)dx/dt + g(x) = 0$ has more than one periodic solution. This disproves a theorem stated by Serbin [same Quart. **8**, 296-303 (1950), Theorem II; these Rev. **12**, 181]. *P. Hartman* (Baltimore, Md.).

Sansone, Giovanni. Soluzioni periodiche dell'equazione di Liénard. Calcolo del periodo. Univ. e Politecnico Torino. Rend. Sem. Mat. **10**, 155-171 (1951).

The existence and uniqueness of periodic solutions of Liénard's equation $\ddot{x} + \omega f(\dot{x}) + \omega f(\dot{x})di/dt + \omega^2 x = 0$, where ω is a positive constant, is studied under milder conditions than in a previous paper by the author [Ann. Mat. Pura Appl. (4) **28**, 153-181 (1949); these Rev. **12**, 260]. It is proved that a unique periodic solution exists, if $f(\dot{x})$ satisfies the following conditions: It is continuous in $(-\infty, \infty)$, negative in a finite interval containing the origin in its interior, positive outside this interval, non-increasing for negative \dot{x} , non-decreasing for positive \dot{x} , and $|f(\dot{x})| < 2$. This solution is stable in the sense that the corresponding limit cycle in the $(\dot{x}, di/dt)$ -plane is stable. Next, the author shows that an existence theorem for periodic solutions proved in his previous paper remains true, if $f(\dot{x})$ is permitted to have a discrete set of jump-discontinuities. If, in addition, this periodic solution is unique, the author proves a comparison theorem which states, roughly, that the limit cycle shrinks, if $f(\dot{x})$ is increased. The last two theorems make possible an iterative procedure for the numerical calculation of the periodic solution. In this method $f(\dot{x})$ is replaced by an approximating step function. For the simplified differential equation a sequence of initial value problems must then be solved explicitly.

W. Wasow (Los Angeles, Calif.).

Crossley, F. R. Erskine. A hyperelliptic function as a nonlinear oscillation. J. Math. Physics **30**, 214-225 (1952).

The oscillation considered is that of a particle constrained to move on a frictionless elliptic path under the influence of

gravity acting parallel to the major axis. The solution is expressed in terms of hyperelliptic integrals, and these are evaluated numerically for various parameter values. The phase plane representation of the solutions is discussed. Tables and graphs are given. *E. Pinney.*

Stöhr, Alfred. Über die Differentialgleichungen eines dynamischen Weltmodells. I. Math. Nachr. **6**, 71-88 (1951).

The author studies the differential system

$$|B|\ddot{B} + k\dot{B} = 0,$$

where B is a time-dependent $n \times n$ matrix, $|B|$ its determinant, and k a constant. In the case $n=3$, applications are given to the cosmological models of Milne and of Heckmann.

A. Schild (Pittsburgh, Pa.).

Kuntzmann, J., Daniel, J., et Ma, Min-Yuan. Stabilité des systèmes de réglage. Méthodes d'étude. Rev. Gén. Électricité **61**, 149-152 (1952).

A brief expository article on methods for investigating the stability of linear control systems. Of the various methods discussed, those which are at all out of the ordinary seem to be still in an undeveloped state, and to be of unproved value. *L. A. MacColl* (New York, N. Y.).

Lueg, R., Päsler, M., und Reichardt, W. Das Impulsintegral, ein Gegenstück zum Duhamelschen Stossintegral. Ann. Physik (6) **9**, 307-315 (1951).

The authors consider a system consisting of a linear ordinary differential equation with constant coefficients, of order n , and prescribed initial conditions. Using the Laplace transformation in a strictly formal manner they observe that the solution of that system can be written in terms of a convolution integral involving the nonhomogeneous terms and solutions of the system when those terms are replaced by impulse functions. The Duhamel form of the solution can be derived in the same manner without introducing impulse functions. Slight modifications of that derivation are made here to get the formula involving impulses, from which the authors note some formal rules concerning Fourier transforms of the solution. *R. V. Churchill.*

Coddington, Earl A., and Levinson, Norman. A boundary value problem for a nonlinear differential equation with a small parameter. Proc. Amer. Math. Soc. **3**, 73-81 (1952).

The solution $y(x, \epsilon)$ of the boundary value of problem (1): $\epsilon y'' + f(x, y)y' + g(x, y) = 0$, $y(0) = y_0$, $y(1) = y_1$, for small $\epsilon > 0$, is compared with the solution $u(x)$ of the initial value problem (2): $f(x, u)u' + g(x, u) = 0$, $u(1) = y_1$, on the interval I : $0 \leq x \leq 1$. It is assumed that (a) problem (2) has a solution $u(x)$ on I , with $u(0) = u_0 \geq y_0$; (b) $f(x, y)$, $g(x, y)$ are of class C' in a region R : $0 \leq x \leq 1$, $|y - u(x)| \leq a$ ($a > 0$); (c) $f(x, y) \geq k > 0$ in R . It is then shown that problem (1) has, for small $\epsilon > 0$, a solution $y(x, \epsilon)$ in R , and that, as $\epsilon \rightarrow 0$, $y(x, \epsilon) \rightarrow u(x)$, $y'(x, \epsilon) \rightarrow u'(x)$, uniformly on any interval $\delta \leq x \leq 1$, $\delta > 0$. The last statement, with slightly different assumptions, but assuming the existence of $y(x, \epsilon)$, was proved by R. v. Mises [Acta Sci. Math. Szeged **12**, Pars B, 29-34 (1950); these Rev. **12**, 101]. It is further proved that the solution $y(x, \epsilon)$ is unique in a region $R_0 \subseteq R$, defined like R but with a smaller value of the constant a .

G. E. H. Reuter (Manchester).

Toso, Annamaria. A proposito di un problema al contorno per equazioni differenziali ordinarie del terzo ordine. Rend. Sem. Mat. Univ. Padova 20, 299-306 (1951).

Let $f(x, y, u, v)$ be real and continuous for $a \leq x \leq b$ and all y, u, v , and suppose that $0 < m \leq f \leq M < \infty$. If α_i ($i=1, \dots, 4$) are real numbers and $a \leq x_i \leq b$, then there exist at least one real number λ_0 and one function $y_0(x)$ (obtained as a fixed point of a functional transformation) such that $y_0(x_i) = \alpha_i$ and $y_0'''(x) = \lambda_0 f(x, y_0(x), y_0'(x), y_0''(x))$ for $a \leq x \leq b$.
F. A. Ficken (Knoxville, Tenn.).

Fishel, B. The continuous spectra of certain differential equations. J. London Math. Soc. 27, 175-180 (1952).

It has been shown by Hartman and Wintner [Amer. J. Math. 71, 214-217 (1949); these Rev. 10, 455] that if q is continuous for $0 \leq t < \infty$ and satisfies $q(t) < \text{const.}$, and if (1) $x'' + (\lambda + q)x = 0$ has a solution satisfying $x(t) = O(1)$, as $t \rightarrow \infty$, then no solution linearly independent of $x(t)$ is of class $L^2(0, \infty)$. Fishel states that the condition " $q(t) < \text{const.}$ " can be replaced by any of the following: (I) q is of class $L^2(0, \infty)$; (II) $q > 0$ and $q(t_2) - q(t_1) < M(t_2 - t_1)$ when $t_1 < t_2$; (III) $|q(t_2) - q(t_1)| < M|t_2 - t_1|$. Although this assertion is correct, the author's proof in the cases (II), (III) is not valid; the error occurs in an integration on p. 178. The author claims that $x = O(1)$ implies $x' = O(1)$ in cases (I)-(III); the example $q = t$ shows that this is false in cases (II), (III). The proofs can be shortened by an appeal to a result of Hartman and Wintner [Amer. J. Math. 72, 148-156 (1950), p. 150; these Rev. 12, 179].

P. Hartman (Baltimore, Md.).

Titchmarsh, E. C. Travaux récents sur la théorie des fonctions caractéristiques. Bull. Soc. Roy. Sci. Liège 20, 543-562 (1951).

This article is an expository one dealing with the relationships between the behavior of $q(t)$ at $t = \infty$ (or $t = \pm \infty$) and the spectra of self-adjoint boundary value problems associated with $x'' + q(t)x = 0$, where $q(t)$ is continuous for $0 \leq t < \infty$ (or $-\infty < t < \infty$).

P. Hartman.

Livšic, M. S. On the theory of self-adjoint systems of differential equations. Doklady Akad. Nauk SSSR (N.S.) 72, 1013-1016 (1950). (Russian)

The author uses his theory of characteristic matrix-functions [Mat. Sbornik 26(68) 247-264 (1950); these Rev. 11, 669] and the results of Potapov on matrix-functions [see same Doklady 72, 849-852 (1950); these Rev. 13, 737] to find necessary and sufficient conditions for an unbounded Hermitian operator A to be unitarily equivalent to a differential operator Df , one of the forms for which is

$$Df = iQ(x) \frac{d}{dx} Q(x) f(x), \text{ where } \int_0^1 |Q(x)|^{-2} dx < \infty,$$

f being a vector- and Q a matrix-function. It is required that A have finite deficiency indices and that A should have an extension B with $A \subset B \subset A^*$, with no spectrum and with "non-negative imaginary part" i.e. such that $\text{Im}(Bf, f) \geq 0$.

F. V. Atkinson (Ibadan).

Urabe, Minoru. On solutions of the linear homogeneous partial differential equation in the vicinity of the singularity. I, II, III. J. Sci. Hiroshima Univ. Ser. A. 14, 115-126, 195-207 (1950); 15, 25-37 (1951).

These papers deal with the equation

$$(*) \quad Xf = \sum_{i=1}^n X_i \frac{\partial f}{\partial x_i} = 0, \\ X_i = \sum_{j=1}^n a_{ij} x_j + \sum_{k_1+\dots+k_n \geq 2} c_{k_1 \dots k_n} x_1^{k_1} \dots x_n^{k_n}.$$

Without loss of generality the matrix $A = (a_{ij})$ is taken in the Jordan normal form

$$A = \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{12} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & A_{ss} \end{pmatrix}$$

where A_{ij} is an n_{ij} by n_{ij} matrix with λ_i 's on the main diagonal and 1's immediately above. Two conditions on the λ_i are considered. C_m : There is a convex domain in the complex plane containing $\lambda_1, \dots, \lambda_m$, but not containing 0. D_m : For all sets of m non-negative integers, p_1, \dots, p_m , if $p_1 + \dots + p_m \geq 2$ then $p_1 \lambda_1 + \dots + p_m \lambda_m \neq \lambda_i$, $i=1, \dots, s$.

In I both C_m and D_m are assumed with $m=s$. Let the independent variables be reindexed so $x_{i,k}$ is the k th variable corresponding to the block A_{ij} in A . For each A_{ij} , n_{ij} functions $\varphi_{i,j,0}, \dots, \varphi_{i,j,n_{ij}-1}$, are constructed so that $X\varphi_{i,j,k} = \lambda_i \varphi_{i,j,k} + k\varphi_{i,j,k-1}$ and $\varphi_{i,j,k} = k! x_{i,n_{ij}-k} + \dots$ a power series in the x 's starting with terms of the second degree. Let $U_{i,j,0}, \dots, U_{i,j,n_{ij}-2}$ be defined by $\varphi_{i,j,k+1} = \sum_{i=0}^k C_i U_{i,j,k} \varphi_{i,j,k-i}$. The functions $(\log \varphi_{i,j,0})/\lambda_i - U_{i,j,0}, U_{i,j,1}, \dots, U_{i,j,n_{ij}-2}$ are integrals of (*). Together with the functions $(\varphi_{i,j,0})^{1/\lambda_i} / (\varphi_{1,0})^{1/\lambda_1}$ (i and j not both 1) they form a set of $n-1$ independent integrals of (*).

In II C_m and D_m are assumed with $m < s$. Let x_1, \dots, x_s be the variables corresponding to the blocks A_{ij} with $i \leq m$. Power series in x_1, \dots, x_s starting with terms of the second degree are constructed to satisfy

$$\sum_{\beta=1}^m X_\beta(x_1, \dots, x_\beta, u_{\beta+1}, \dots, u_n) \frac{\partial u_\alpha}{\partial x_\beta} \\ = X_\alpha(x_1, \dots, x_\beta, u_{\beta+1}, \dots, u_n), \quad \alpha = \mu+1, \dots, n.$$

Let S be the set of all $x = (x_1, \dots, x_s)$ such that

$$x_\alpha = u_\alpha(x_1, \dots, x_\beta), \quad \alpha = \mu+1, \dots, n.$$

There are $n-1$ independent functions $g_\sigma(x_1, \dots, x_\beta)$, $\sigma=1, \dots, \mu-1$, and

$$g_\alpha(x_1, \dots, x_\beta) = x_\alpha - u_\alpha(x_1, \dots, x_\beta), \quad \alpha = \mu+1, \dots, n$$

which satisfy (*) when $x \in S$. Any solution of (*) is equal on S to a function of $g_1, \dots, g_{\mu-1}$.

Severe difficulties can arise if D_m does not hold. Nevertheless, in III similar results are obtained assuming only C_m (or only $C_m, m < s$). F. M. Stewart (Providence, E. I.).

Urabe, Minoru. On integrals of the certain ordinary differential equations in the vicinity of the singularity. I, II. J. Sci. Hiroshima Univ. Ser. A. 14, 209-215 (1950); 15, 39-43 (1951).

The results in the papers reviewed above are applied to the system

$$(**) \quad \frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \dots = \frac{dx_n}{X_n}.$$

Using the notations of the preceding review let

$$\nu_i + 1 = \max(n_{i1}, n_{i2}, \dots).$$

If C_s and D_s both hold then, in some neighborhood of the origin, every solution of (***) can be expanded as a power series in the functions $t^{i_1}(\log t)^{j_1}$, $0 \leq i_1 \leq s$, $0 \leq j_1 \leq \nu_{i_1}$, where t is a suitably chosen parameter. With a modified definition of ν_i , this remains valid if only C_s is assumed.

F. M. Stewart (Providence, R. I.).

Kasuga, Takashi. Generalisation of R. Baire's theorem on differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} f(x, y) = 0.$$

Proc. Japan Acad. 27, 117-121 (1951).

Under suitable regularity assumptions a solution $z(x, y)$ of the equation (1) $z_x + f(x, y)z_y = 0$ will be constant along the characteristic curves given by $dy/dx = f(x, y)$. This theorem is proved here assuming only that $f(x, y)$ and $f_y(x, y)$ are continuous in a domain G , and that $z(x, y)$ is continuous in G and possesses derivatives z_x, z_y except at a denumerable set of points of G , (1) being satisfied almost everywhere.

F. John (New York, N. Y.).

Mendes, Marcel. Sur un système d'équations aux différentielles totales généralisant les équations canoniques. C. R. Acad. Sci. Paris 234, 1665-1667 (1952).

The author generalizes the idea of a Hamiltonian canonical system by introducing $r > 1$ independent variables. We suppose that $H_s = H_s(p_1, \dots, p_n, q_1, \dots, q_n, t_1, \dots, t_r)$ with $k = 1, \dots, r$. The equations

$$\partial q_i / \partial t_k = \partial H_s / \partial p_i, \quad \partial p_i / \partial t_k = -\partial H_s / \partial q_i, \quad i = 1, \dots, n,$$

are then supposed to be completely integrable, and the author shows that the usual properties of Hamiltonian equations admit appropriate generalizations. Such, for example, are the classical results of Poincaré and E. Cartan on the equations of variation and integral invariants. He also obtains a transformation to a generalized Lagrangian system.

D. C. Lewis (Baltimore, Md.).

Bechert, Karl. Lösungen und Lösungsverfahren für nicht-lineare partielle Differentialgleichungen. Math. Nachr. 6, 271-292 (1952).

The paper is concerned with instances, where the solution of a partial differential equation can be achieved by introducing the first derivatives of the unknown function as new independent variables. Applying this idea to a non-linear first order equation of the form $F(u, u_{x_1}, \dots, u_{x_n}) = 0$, the author obtains the general solution neatly in terms of quadratures. In case F does not depend on u , the solution is obtained explicitly even without quadratures. These results are simpler than those obtained by the ordinary method of characteristics. The same idea is applied to the equations of 1-dimensional gas dynamics, but the results here do not appear to differ essentially from those obtained ordinarily by the Legendre transformation.

F. John.

Beckert, Herbert. Die Abhängigkeit der Lösungen quasilinearer elliptischer Systeme partieller Differentialgleichungen erster Ordnung mit zwei unabhängigen Variablen von einem Parameter. Math. Nachr. 5, 111-121 (1951).

A perturbation problem is discussed for real solutions u, v of an elliptic system of two quasilinear equations taken

without loss of generality in complex form as

$$\lambda(x, y, u, v) [p(x, y, u, v)u_x + q(x, y, u, v)v_x] + \bar{p}(x, y, u, v)u_y + \bar{q}(x, y, u, v)v_y = f(x, y, u, v),$$

where λ, p, q, f are complex-valued functions of class C^1 . Let $u_0(x, y), v_0(x, y)$ comprise a known solution which is continuous in the closure of the unit disk E and of class $C_{1+\lambda}$ in E and for which the linear relation $\alpha_0 u_0 + \beta_0 v_0 = \gamma_0$ holds on the boundary S of E . It is assumed that γ_0 is a function of arc length s of class $C_{1+\lambda}$ on S and that α_0, β_0 are functions defined and of class $C_{1+\lambda}$ in $E+S$ and, in addition, that $\alpha_0^2 + \beta_0^2 > 0$ in $E+S$. The problem is to obtain a solution $u(x, y), v(x, y)$, continuous in $E+S$ and of class C^1 in E and satisfying on S the perturbed boundary condition $[\alpha_0 + \epsilon(\alpha - \alpha_0)]u + [\beta_0 + \epsilon(\beta - \beta_0)]v = \gamma_0 + \epsilon(\gamma - \gamma_0)$, where the functions α, β, γ are subjected to the same restrictions as $\alpha_0, \beta_0, \gamma_0$, respectively. A. Douglis (New York, N. Y.).

Beckert, Herbert. Über lineare elliptische Systeme partieller Differentialgleichungen erster Ordnung mit zwei unabhängigen Variablen. Math. Nachr. 5, 173-208 (1951).

A class of boundary problems is discussed for a system of equations of the form

$$u_{i,x} = \sum_{k=1}^{2q} [a_{ik}(x, y)u_{k,x} + b_{ik}(x, y)u_k] + f_i(x, y), \quad i = 1, \dots, 2q$$

($u_{k,x} = \partial u_k / \partial x$, etc.), which, in a domain G that contains the closure of the unit disk $E: x^2 + y^2 < 1$, is elliptic in the following sense: the elementary divisors of the matrix $(a_{ik}(x, y))$ are all non-real in G and all simple. The leading coefficients $a_{ik}(x, y)$ are assumed to be class $C_{1+\lambda}$ (i.e., to have continuous first partial derivatives that satisfy a Hölder condition with exponent λ , $0 < \lambda < 1$) and the other coefficients to be of class C_λ (Hölder-continuous with exponent λ) in G . The problem of determining continuously differentiable functions $u_k(x, y)$ which in E satisfy the $2q$ differential equations and on S satisfy q prescribed linear relations of a certain admitted type is reduced to solving a system of linear integral equations. A. Douglis (New York, N. Y.).

Bergman, S., and Schiffer, M. Kernel functions and partial differential equations. I. Boundary value problems in the theory of non-linear partial differential equations of elliptic type. J. Analyse Math. 1, no. 2, 375-386 (1951). (Hebrew summary)

In this paper the authors describe a method for the solution of the boundary value problem for non-linear partial differential equations which are the Euler-Lagrange equations of suitable variational problems. If $u(t)$ is a known solution of the original equation depending on a parameter t , $v = \partial u / \partial t$ will satisfy a linear partial differential equation, the Jacobi equation of the variational problem. The basic idea of the paper is to use the well-developed theory of the thus arising linear equation for the study of problems related to the original non-linear equation.

This program is carried through in detail in the case of the equation (*) $u_{xx} + u_{yy} = F(x, y, u)$, where F is assumed to be twice continuously differentiable in a domain D bounded by a smooth curve C , and $F(x, y, 0) = 0$, $F_u(x, y, u) > 0$. The Jacobi equation is in this case (**) $v_{xx} + v_{yy} = F_u(x, y, u)v$. If $f(s)$ (s the length parameter on C) are the given boundary values on C , a suitable one-parameter family of solutions $u(t)$ of (*) is provided by solving (**) with the boundary values $tf(s)$; the corresponding boundary values of $v = \partial u / \partial t$

are $f(s)$. Then v can be written down explicitly by means of the Green's function G of (**), and one thus obtains one relation between u_i and G (the latter, of course, also depends on u). Another relation between these quantities is obtained by calculating the variation of G resulting from the change in the coefficient of (**) if t is replaced by $t + \delta t$. The authors thus arrive at a system of two integro-differential equations for u and G , and they solve this system by means of a successive approximations procedure. *Z. Nehari.*

Bergman, S., and Schiffer, M. A majorant method for non-linear partial differential equations. Proc. Nat. Acad. Sci. U. S. A. 37, 744-749 (1951).

Let D be a finite simply-connected domain in the (x, y) -plane bounded by a smooth curve C , and let the boundary values $\mu(s)$ (s being the length parameter on C) be given on C . The authors describe a method of constructing a solution of the non-linear partial differential equation (1) $u_{xx} + u_{yy} = P(x, y)u + \sum_{r=1}^n a_r(x, y)u^r$ ($P > 0$, P and u_r continuous in D , a_r restricted by certain boundedness assumptions) which takes the values $\mu(s)$ on C . The basic idea is to introduce a solution $u(x, y; t)$ of (1) with the boundary values $t\mu(s)$ on C and to expand this solution into a power series (2) $u(x, y; t) = \sum_{r=1}^{\infty} u_r(x, y)t^r$. The coefficients u_r can be successively determined by solving boundary value problems for the linear equation (3) $u_{xx} + u_{yy} - Pu = 0$ or inhomogeneous equations related to it. In fact, if the u_r are made subject to the boundary conditions $u_1 = \mu(s)$, $u_r = 0$, $r > 1$, on C , all coefficients u_r can be expressed in terms of the Green's function of (3) for the domain D .

Everything depends, of course, on the convergence of the series (2). This problem is treated by means of a majoration procedure which is similar to the Cauchy-Kowalevsky process insofar as it replaces the functions $a_r(x, y)$ by their given upper bounds, but departs radically from it by introducing a comparison problem for a larger domain and for constant boundary values $A = \max |\mu(s)|$. The authors succeed in constructing a majorizing problem which can be solved explicitly and guarantees the solution of the original problem provided A is sufficiently small. *Z. Nehari.*

Višik, M. I. On the stability of solutions of boundary problems for elliptic differential equations (relative to variation of the coefficients and right-hand sides). Doklady Akad. Nauk SSSR (N.S.) 81, 717-720 (1951). (Russian)

Consider the elliptic equation

$$Lu = -\sum \partial_i(a_{ik}(x)\partial_k u(x)) + \sum b_i(x)\partial_i u(x) + c(x)u(x) = h(x),$$

where $x \in D$, $\partial_i = \partial/\partial x_i$, with the boundary condition

$$\partial u(s)/\partial \nu - \frac{1}{2} \sum \partial_i a_{ik}(x) + A u(s) = \varphi(s), \quad s \in \Gamma,$$

where D is an open bounded subset of real n -space with smooth boundary Γ , and $\partial/\partial \nu = \sum a_{ik} \cos(\nu, x_k) \partial_k$ is the interior normal derivative. The coefficients of L are supposed to be real and regular enough, A is a bounded linear and non-negative operator on $L_2(\Gamma)$ and it is assumed that $c(x) - \frac{1}{2} \sum \partial_i b_i(x) > 0$. Then the boundary value problem has a unique solution (in a generalized sense) for h in $L_p(D)$, ($p \geq 2n/(n+2)$ when $n > 2$ and $p > 1$ when $n = 2$), and φ in $L_q(\Gamma)$, ($q^* > 2(n-1)/n$), and the solution is a continuous function of h , φ , the coefficients of L and the operator A in the sense that changes of these elements, small with respect to the norms $\|h\|_p$, $\|\varphi\|_{q^*}$, $\sup(|a_{ik}(x)|, |b_i(x)|, |c(x) - \frac{1}{2} \sum \partial_i b_i(x)|)$ and the operator norm $\|A\|$, will produce a change in the solution which is small with respect to the

norm $(\int_D \sum (\partial_i u(x))^2 + u(x)^2 dx)^{1/2}$. The theorem has an extension to the case when A is not bounded and a proof is supplied when $b_i = 0$, $p = q^* = 2$ and A is self-adjoint.

L. Gårding (Lund).

Stampacchia, Guido. Problema di Dirichlet e proprietà qualitative della soluzione. Giorn. Mat. Battaglini (4) 4(80), 226-237 (1951).

Let D be a bounded domain in the (x, y) -plane whose boundary Γ is a simple closed curve of certain differentiability properties, and ϕ a continuous function defined on Γ . The paper is concerned with existence and uniqueness questions concerning "almost everywhere" solutions of the elliptic boundary value problem

$$(1a) \quad a_{11}r + 2a_{12}s + a_{22}t + a_{13}p + a_{23}q = f(x, y),$$

$$(1b) \quad u = \phi \text{ on } \Gamma.$$

Here $r = u_{xx}$, \dots , $q = u_{yy}$; the a_{ik} are functions of x, y , and $a_{11}a_{22} - a_{12}^2$ is bounded away from zero. Under certain assumptions about ϕ one of the results of the paper is that the following conditions are sufficient for (1) to have one and only one "almost everywhere" solution $u(x, y)$ which satisfies the Hölder condition for every exponent $\alpha < 1$, whose derivatives are absolutely continuous (separately with respect to each variable), and whose second derivatives are square integrable over D . Further results concern the approach of u_x and u_y to the boundary Γ and the square integrability over Γ of the limit functions obtained.

E. H. Rothe (Ann Arbor, Mich.).

Caccioppoli, Renato. Limitazioni integrali per le soluzioni di un'equazione lineare ellittica a derivate parziali. Giorn. Mat. Battaglini (4) 4(80), 186-212 (1951).

Let D be a bounded domain in the real Euclidean n -space of points $x = (x_1, x_2, \dots, x_n)$, and S its boundary. Let $A_{ik} = A_{ki}$, $B_i(i, k = 1, \dots, n)$, C and f be functions of x , and \bar{u} a function defined on S . Finally, let u be a solution of the elliptic boundary value problem

$$(1a) \quad \sum_{i,k=1}^n A_{ik} p_{ik} + \sum_{i=1}^n B_i \phi_i + Cu = f \quad (p_i = \partial u / \partial x_i, p_{ik} = \partial^2 u / \partial x_i \partial x_k)$$

$$(1b) \quad u = \bar{u} \text{ on } S,$$

where the quadratic form with the coefficients A_{ik} is positive definite and its discriminant equals 1.

The main object of the paper is to establish (under proper assumptions about S , \bar{u} , and the coefficients of (1a) estimates concerning the integrals

$$(2) \quad I_D = \int_D u^2 d\tau, \quad I'_D = \int_D \sum_{i,k=1}^n p_{ik}^2 d\tau, \quad I''_D = \int_D \sum_{i,k=1}^n p_{ik}^2 d\tau.$$

To state some of the results we first introduce the following notations: I_S , I'_S , I''_S are the integrals obtained from (2) by replacing D by S , and u by \bar{u} . Furthermore, we set $\|u\|_D = I_D + I'_D + I''_D$, $\|\bar{u}\|_S = I_S + I'_S + I''_S$. Then an estimate of the form

$$(3) \quad \|u\|_D \leq \text{const.} \times \left[\|\bar{u}\|_S + I_D + \int_D f^2 d\tau \right]$$

holds. If the uniqueness theorem holds for the boundary problem (1) (e.g. if $C \leq 0$), then (3) holds with I_D deleted. Especially in the cases $n = 2, 3$ it is proved that

$$\|u\|_D \leq \text{const.} \times \left[\|u\|_S^* + \int_D f^2 d\tau \right]$$

where $\|u\|_D^*$ and $\|\tilde{u}\|_S^*$ are obtained from $\|u\|_D$ and $\|\tilde{u}\|_S$ by replacing I_D by $\max_D u^2$ and I_S by $\max_S \tilde{u}^2$. In the latter case estimates for the Hölder constants of u are also given.

E. H. Rothe (Ann Arbor, Mich.).

Kapilevič, M. B. On Cauchy's problem for the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial \sigma^2} - \frac{a}{\sigma} \frac{\partial u}{\partial \sigma} - b^2 u = 0.$$

Doklady Akad. Nauk SSSR (N.S.) 81, 13-16 (1951). (Russian)

Kapilevič, M. B. On an equation of mixed elliptic-hyperbolic type. Mat. Sbornik N.S. 30(72), 11-38 (1952). (Russian)

The equation

$$(*) \quad \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_{n-1}^2} + \frac{1}{y^p} \frac{\partial^2 u}{\partial y^2} - b^2 u = 0$$

with $(-1)^p = -1$ is elliptic for $y > 0$, hyperbolic for $y < 0$ and parabolic for $y = 0$. The Dirichlet and Neumann problems are solved for the half-space $y > 0$ with data prescribed on the hyperplane $y = 0$. For the case $n = 2$ the Dirichlet problem is solved for a domain bounded by a "normal" curve, i.e. a curve of the form $(2+p)^2 x^2 + 4y^{p+2} = (2+p)^2$, and the segment of the x_1 -axis that this curve intersects. For the cases $n = 2, 3$ the Cauchy problem is solved for the hyperbolic domain with the data given on the parabolic line (plane). The methods involve reducing (*) to an equation in which the second order operator is the Laplacian in the elliptic case and the wave-operator in the hyperbolic case. Then by separation of variables and the use of special functions integral formulas for the solutions are obtained. In the elliptic case asymptotic formulas are given for the solution as $R^2 = \sum_{i=1}^n x_i^2 \rightarrow \infty$ and in the hyperbolic case ($n = 2$) a formula is given for the Riemann function. An application is given to equations of the form $u_{xx} - u_{\sigma\sigma} - (K)^{-1} K_{\sigma} u_{\sigma} = 0$ which occur in gas dynamics and plasticity.

M. H. Protter (Princeton, N. J.).

Ingersoll, Benham-M. Problèmes pour les équations hyperboliques avec des conditions initiales sur les dérivées supérieures. C. R. Acad. Sci. Paris 234, 693-694 (1952).

The classical Cauchy problem concerning the hyperbolic equation

$$(1) \quad u_{xx} + au_x + bu_y + cu = f$$

and a curve C consists in finding a solution of (1) for which $u = u_{00}$ and $u_{\sigma} = u_{10}$ are prescribed on C . For "general position" of C this problem has a unique solution in a certain rectangle. The present note discusses the problem obtained by replacing the couple u_{00}, u_{10} by the couple $u_{ij}, u_{ij\sigma}$ where for any couple i, j of non-negative numbers $u_{ij} = \partial^i u / \partial x^i \partial y^j$.

E. H. Rothe (Ann Arbor, Mich.).

Bureau, Florent. Le problème de Cauchy et les séries de fonctions fondamentales. C. R. Acad. Sci. Paris 234, 791-792 (1952).

The author's aim is to give a connecting link between the integral representations and series expansions encountered in solving boundary value problems for hyperbolic equations. Let $u(x, t)$ be a solution of

$$\frac{\partial^2 u}{\partial t^2} - L_x(u) = 0 \quad \text{for } 0 < x < l, \quad t > 0,$$

$$u(0, t) = u(l, t) = 0, \quad u(x, 0) = u_0(x), \quad \left(\frac{\partial u}{\partial t} \right)_{t=0} = u_1(x),$$

where L_x is a self-adjoint second order differential operator. Denoting by $v_k(x)$ the eigen-functions of L_x belonging to the eigen-value λ_k for 0 boundary values, the function

$$\mathcal{K}_s(x, y; t) = \sum_{k=1}^{\infty} \frac{v_k(x)v_k(y) \sin(t\sqrt{\lambda_k})}{\lambda_k^{s+1}}$$

is introduced. Then the solution $u(x, t)$ is given by the analytic continuation of the expression

$$- \int_0^t \left[u_s(y) \mathcal{K}_s(x, y; t) + u_0(y) \frac{\partial \mathcal{K}_s(x, y; t)}{\partial t} \right] dy$$

for $s = 0$. Here $2\mathcal{K}_{s=0}(x, y; t)$ reduces to the Riemann function $R(x, t; y, 0)$ belonging to the point (x, t) for y in the interval $(x-t, x+t)$, and vanishes for y outside that interval.

F. John (New York, N. Y.).

Pini, Bruno. Sulle equazioni a derivate parziali, lineari del secondo ordine in due variabili, di tipo parabolico. Ann. Mat. Pura Appl. (4) 32, 179-204 (1951).

Let D be the domain $a \leq y \leq b$, $\chi_1(y) \leq x \leq \chi_2(y)$, where the functions $\chi_1(y) < \chi_2(y)$ have continuous first derivatives, and consider the differential equation

$$(1) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} + \alpha(x, y)u = f(x, y).$$

Here $\alpha(x, y)$ is assumed continuous on D and satisfies a Hölder condition with respect to y . The function $f(x, y)$ has these same properties interior to D and $|f|^r$ ($r > 3/2$) is summable over D . The following generalized type of boundary-value problem is the main result of the paper. If $|f_1(y)|^r$ and $|f_2(y)|^r$ are summable over $a \leq y \leq b$ with $1 < r < 2$, then there exists one and only one function $u(x, y)$ such that $u(x, b) = 0$ for $\chi_1(b) \leq x \leq \chi_2(b)$, is a regular solution of (1) interior to D , and satisfies the boundary conditions

$$\lim_{t \rightarrow 0} \int_a^b |u(\chi_1(y) - (-1)^i t, y) - f_i(y)|^r dy = 0, \quad i = 1, 2.$$

F. G. Dressel (Durham, N. C.).

Fulks, W. On the solutions of the heat equation. Proc. Amer. Math. Soc. 2, 973-979 (1951).

In his unpublished thesis [University of Michigan, 1949], the author obtained independently some results given by Hartman and Wintner [Amer. J. Math. 72, 367-395 (1950); these Rev. 12, 104]. These results include (1) necessary and sufficient conditions for the representability of solutions u of the heat equation $u_{xx} = u_t$ in a rectangle as a Stieltjes integral over three sides of the boundary of the rectangle, and (2) formulae for the weight functions, when such a representation is possible. In view of the uniqueness of the representation, when it exists, these results give, as a special case, theorems on the representability of u as a Lebesgue integral; all that is involved is the absolute continuity of the weight functions [cf. Hartman and Wintner, loc. cit., remark on p. 376]. In the present paper, the author deduces, from standard theorems in integration theory and from the main theorem on the representation of u in terms of Stieltjes integrals, other forms of the necessary and sufficient conditions that u be representable as a Lebesgue integral.

P. Hartman (Baltimore, Md.).

Pignedoli, Antonio. Su un problema di diffusione della Fisica-matematica. Ann. Mat. Pura Appl. (4) 32, 281-293 (1951).

Let D, c, h, Λ be constants, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, S a region in space with boundary σ on which n denotes the

exterior normal. While the author starts out with the boundary-value problem of finding a solution $\rho = \rho(x, y, z, t)$ of $D\Delta\rho - \rho_t + c\rho = g(x, y, z, t)$ in S and $\partial\rho/\partial n + h\rho = 0$ on σ , his assumptions on ρ and g are such as to reduce the problem to finding a solution $u = u(x, y, z)$ of $\Delta u + \lambda u = f(x, y, z)$ in S and $\partial u/\partial n + hu = 0$ on σ . The procedure for replacing this problem by an integral equation, through introducing a Green function, is sketched. For the special case S is a sphere, and the given function f can be expanded in terms of surface spherical harmonics, the above boundary value problem is solved by the method of separation of variables.

F. G. Dressel (Durham, N. C.).

Ossicini, Alessandro. Il calcolo simbolico e la propagazione del calore in una ipersfera dello spazio euclideo ad n dimensioni. Ann. Scuola Norm. Super. Pisa (3) 5, 269-278 (1951).

Under the assumption of spherical symmetry the parabolic equation $u_{x_1x_1} + \dots + u_{x_nx_n} - \alpha u_t = 0$ takes the form

$$(1) \quad \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r} \frac{\partial u}{\partial r} - \alpha \frac{\partial u}{\partial t} = 0.$$

Here α is a positive constant and $r^2 = \sum_{i=1}^n x_i^2$. A finite Hankel transform is used in solving the following heat conduction problem. Find a function $u(r, t)$ which is a solution of (1) in the sphere $r < 1$, and which in addition meets the three conditions $u(r, 0) = F(r)$, $u_r(1, t) + \lambda u(1, t) = 0$, $\lim_{t \rightarrow \infty} u(r, t) = 0$. The given continuous function $r^{1-n}F(r)$ is assumed to be of bounded variation, and λ is a constant.

F. G. Dressel (Durham, N. C.).

Plesset, M. S., and Zwick, S. A. A nonsteady heat diffusion problem with spherical symmetry. J. Appl. Phys. 23, 95-98 (1952).

The problem considered is that of heat diffusion outside an expanding or contracting spherical bubble surrounded by a nonviscous incompressible thermally homogeneous liquid. Assuming the temperature gradient on the surface of the bubble is known, the solution is found near the surface by a process of successive approximations. A first approximation (called a "zero-order solution") is given completely, and a second approximation (called a "first-order correction") is given partially. Assuming that the temperature is essentially constant except near the bubble surface, the authors apply boundary conditions "at infinity" in spite of the fact that the solution is restricted to the region near the bubble surface. It is clear that this assumption implies significant restrictions on the bubble surface temperature gradient (assumed to be a given function of time), but this question is not discussed. E. Pinney (Berkeley, Calif.).

Montroll, Elliott W., and Newell, Gordon F. Unsteady-state separation performance of cascades. I. J. Appl. Phys. 23, 184-194 (1952).

The mathematical problem is as follows: Find a solution $c(n, T)$, $0 \leq n \leq N$, $0 \leq T < \infty$, of the non-linear partial differential equation

$$(*) \quad H(n) \frac{\partial c}{\partial t} = \frac{\partial}{\partial n} \left\{ W(n) \left[\frac{\partial c}{\partial n} - \psi c(1-c) \right] - Pc \right\},$$

where ψ and P are constants, such that $c(0, T) = \text{constant}$, $\partial c/\partial n = \psi c(1-c)$ for $n = N$ and all T , and such that $c(n, 0)$ is given. The authors first discuss the stationary case, $\partial c/\partial T = 0$. Then, they assume W and H constant and linearize (*) by introducing a new dependent variable. This device

is similar to that by which a Riccati equation is changed to a linear second order equation. The problem then reduces to finding a solution of the heat equation $\phi_t = \phi_{xx}$ such that $\phi_{xx} + k\phi_x - \beta\phi = 0$ at $x = a$, $\phi_x = \alpha\phi$ at $x = 0$ and $\phi(x, 0)$ is a given function of x . Here, α, β, k are constants. The authors solve this problem by separation of variables. Some of the manipulations they used could have been avoided by the application of a method due to R. E. Langer in a similar problem [Tôhoku Math. J. 35, 260-275 (1932)].

B. Friedman (New York, N. Y.).

Churchill, R. V. Integral transforms and boundary value problems. Amer. Math. Monthly 59, 149-155 (1952). Expository lecture.

Sbrana, F., and Fumi, F. Errata corrige: Integrazione dell'equazione dei telegrafisti per mezzo degli operatori funzionali. Atti Accad. Ligure 7, 203-206 (1951). See same Atti 6, 273-298 (1950); these Rev. 13, 133.

Garnir, Henri. Sur la transformation de Laplace des distributions. C. R. Acad. Sci. Paris 234, 583-585 (1952).

Dans l'espace R^{n+1} , un point sera représenté par une variable spatiale $x(x_1, \dots, x_n)$ et une variable temporelle t . Soit $T_{x,t}$ une distribution sur R^{n+1} [Schwartz, Théorie des distributions, tomes I, II, Hermann, Paris, 1951; ces Rev. 12, 31, 833] ayant son support dans l'ensemble $t \geq 0$. On suppose que, pour toute fonction $\varphi(\xi, \tau) \in D_{t,\tau}$, la fonction de t : $\exp(p t) T_{x,t} \cdot [\varphi(\xi, t + \tau)]$ soit sommable en t pour $p > p_0$, d'intégrale $\tau_i^{(p)} \cdot [(f)_{t=0}^+ \exp(p \tau) \varphi(\xi, \tau) d\tau]$, $\tau_i^{(p)}$ étant, pour $p > p_0$, une distribution en ξ dépendant continuellement du paramètre p ; alors cette dernière est appelée transformée de Laplace de T par rapport au temps seul. Si alors $L(\partial/\partial x, \partial/\partial t)$ est un opérateur différentiel à coefficients constants, et si $T_{x,t}$ est telle que $\tau_i^{(p)}$ soit, pour $p > p_0$, solution élémentaire de l'opérateur différentiel d'espace $L(\partial/\partial x, p)$, alors T est solution élémentaire de $L(\partial/\partial x, \partial/\partial t)$. Applications à l'équation des ondes à 1, 2, 3 dimensions d'espace.

L. Schwartz (Paris).

Mikusinski, J. G.-. Sur les équations différentielles du calcul opératoire et leurs applications aux équations classiques aux dérivées partielles. Studia Math. 12, 227-270 (1951).

Dans un article antérieur [Drobot et Mikusinski, Studia Math. 11, 38-40 (1949); ces Rev. 12, 9] l'auteur a montré que toute équation différentielle d'ordre n ,

$$(1) \quad a_0 x^{(n)}(\lambda) + a_1 x^{(n-1)}(\lambda) + \dots + a_n x(\lambda) = f(\lambda),$$

pour une fonction $x(\lambda)$ d'une variable réelle, à valeurs dans une algèbre topologique Q sans diviseur de 0, a au plus une solution, dont les dérivées d'ordre $\leq n-1$ aient des valeurs données pour $\lambda = \lambda_0$ (conditions de Cauchy d'ordre n). Ici Q sera le corps précédemment introduit par l'auteur [ibid. 11, 41-70 (1949); ces Rev. 12, 189], corps des fractions de l'algèbre de convolution des fonctions numériques continues sur $(0, +\infty)$. Un élément de Q est dit logarithme si l'équation différentielle $x'(\lambda) - ax(\lambda) = 0$, $x(0) = \text{unité}$, a une solution, alors notée $\exp a\lambda$. L'auteur montre alors ce qui suit: Si l'équation caractéristique de (1): $a_0 w^n + a_1 w^{n-1} + \dots + a_n = 0$ a exactement n racines dans Q , dont p sont logarithmes, l'équation (1) homogène ($f(\lambda) = 0$) a exactement p solutions Q -indépendantes; l'équation non homogène a au plus 1 solution satisfaisant à des conditions de Cauchy d'ordre p ; sa recherche est ramenée à la résolution successive d'une équation "logarithmique" d'ordre p (dont l'équation caractéris-

tique a p racines logarithmiques), qui a toujours une racine et une seule pour des conditions de Cauchy d'ordre p , et d'une équation pure (dont l'équation caractéristique est sans racine logarithmique) qui a 0 ou 1 solution.

L'auteur donne des applications aux équations aux dérivées partielles. L'équation à coefficients constants $\sum_{\alpha} a_{\alpha} x^{\alpha} \partial^{\alpha} u / \partial t^{\alpha} = \varphi(t, \lambda)$ sera considérée comme une équation différentielle en λ pour la fonction $x(\lambda)$ à valeurs dans $Q(t)$. Une équation elliptique donne lieu à une équation pure, une équation hyperbolique à une équation logarithmique. Théorèmes d'existence et d'unicité nombreux et variés.

L. Schwartz (Paris).

Difference Equations, Special Functional Equations

Myškis, A. D. General theory of differential equations with a retarded argument. Amer. Math. Soc. Translation no. 55, 62 pp. (1951).

Translated from Uspehi Matem. Nauk (N.S.) 4, no. 5(33), 99-141 (1949); these Rev. 11, 365.

Nørlund, N. E. Séries hypergéométriques. Kungl. Fysio-grafiska Sällskapet i Lund Föreläsningar [Proc. Roy. Physiol. Soc. Lund] 21, no. 15, 4 pp. (1952).

For the linear difference equation

$$\sum_{i=0}^{n+1} (-1)^i \binom{x-\gamma_0}{i} \left[\Delta Q(x) - \sum_{i=1}^{i-1} \Delta R(x-1) \right] u(x-i) = 0$$

in which x is the variable, γ_0 and z are parameters, and $Q(x)$ and $R(x)$ are polynomials of degree $n+1$ and n respectively, the author gives two fundamental systems of solutions consisting of generalized hypergeometric series with variables z and z^{-1} respectively. He gives two more fundamental systems in which the solutions are infinite series of terminating generalized hypergeometric series of variables z and z^{-1} respectively.

A. Erdélyi (Pasadena, Calif.).

Bagchi, Hari das, and Chatterjee, Phatik Chand. Note on a second functional equation, connected with the function $\varphi(x)$. Amer. Math. Monthly 59, 91-92 (1952).

It is shown that the most general meromorphic solution of the equation $f(x+y) - f(x-y) = -f'(x)f'(y)\{f(x)-f(y)\}^{-2}$ is $f(x) = \varphi(x) + C$, C an arbitrary constant. I. M. Sheffer.

Bagchi, Hari das, and Chatterji, Phatik chand. On a (third) functional equation, connected with the Weierstrassian function $\varphi(z)$. Boll. Un. Mat. Ital. (3) 6, 280-284 (1951).

The authors find that the most general meromorphic solution of the functional equation

$$f(x+y)f(x-y) = \frac{\{f(x)f(y)+a\}^2 + b\{f(x)+f(y)\}}{f(x)-f(y)}$$

(a, b being constants) is $f(z) = \varphi(\mu z)$, where μ is an arbitrary constant and where the invariants g_2, g_3 for φ involve a and b .

I. M. Sheffer (State College, Pa.).

Integral Equations

Satō, Tokui. Détermination unique de solution de l'équation intégrale de Volterra. Proc. Japan Acad. 27, 276-278 (1951).

The note concerns the Volterra integral equation

$$u(x) = f(x) + \int_a^x K(x, t, u(t)) dt$$

where f is regular in a domain D of the x -plane, K is regular in x, t, u , for $|x-a| < l$ and (t, u) in a domain \mathfrak{D} , the integration being along a curve C , which with respect to the point x_0 has the property that for each r , there exists a point x_1 on C such that between x_1 and x_0 , C lies in the circle $|x-x_0| < r$. It is proved that if there exists a unique analytic solution of the equation along C for $x \neq x_0$ and (x_0, u_0) lies in \mathfrak{D} where u_0 is any value approached by u as x approaches x_0 along C , then the solution is also analytic at x_0 and consequently the uniqueness extends to x_0 .

T. H. Hildebrandt (Ann Arbor, Mich.).

Buscham, W. Die Zurückführung von speziellen linearen Integrodifferentialgleichungen auf gewöhnliche Integralgleichungen. Z. Angew. Math. Mech. 32, 20-21 (1952). The equation

$$(1) \quad y(x) - \lambda \int_a^x G(x, \xi) N[y(\xi)] d\xi = f(x),$$

where $N[y(x)] = \sum_{r=0}^n q_r(x) d^r y(x)/dx^r$, and $q_r(x)$, $d^r f(x)/dx^r$, $\partial^r G(x, \xi)/\partial x^r$ ($r=1, \dots, n$) are continuous in $a \leq x \leq b$, is converted (by applying the operator N) into the Fredholm equation

$$(2) \quad \Phi(x) - \lambda \int_a^b H(x, \xi) \Phi(\xi) d\xi = F(x),$$

with $\Phi(x) = N[y(x)]$, $F(x) = N[f(x)]$, $H(x, \xi) = N_x[G(x, \xi)]$. Explicit formulae are derived which express the solution of (1) in terms of the solution of (2). G. E. H. Reuter.

Sunouchi, Gen-ichirō. A class of singular integral equations. Tôhoku Math. J. (2) 3, 220-222 (1951).

The integral equation

$$f(y) = \pi^{-1}(P) \int_{-1}^1 \frac{u(x)}{x-y} dx - \frac{1}{\pi} \int_{-1}^1 k(y, x) u(x) dx, \quad |y| < 1,$$

is transformed into a Fredholm equation, first changing the variables by $x = \cos \phi$, $y = \cos \theta$ and then taking conjugate functions. If the resolvent kernel of the Fredholm equation is known, an explicit formula for $u(x)$ can be derived. The method appears to be a very special case of the 'regularising' technique developed by N. I. Mushelišvili [Singular integral equations . . . , Moscow-Leningrad, 1946; these Rev. 8, 586], S. G. Mihlin [Uspehi Matem. Nauk (N.S.) 3, no. 3(25), 29-112 (1948); these Rev. 10, 305] and others.

G. E. H. Reuter (Manchester).

Tricomi, Francesco G. The airfoil equation for a double interval. Z. Angew. Math. Physik 2, 402-406 (1951).

The integral equation

$$(A) \quad \frac{1}{\pi} \int_{-1}^{-k} + \int_k^1 \frac{\varphi(y)}{y-x} dy = f(x), \quad k < |x| < 1$$

(upper limit $-k$ misprinted as k in original) arises in attempting to extend Prandtl's lifting line theory to swept

wings. The author establishes a solution to (A) in the form

$$(B) \quad \varphi(x) = g(x) - \left\{ \frac{1}{\pi} \int_{-1}^1 [(1-y^2)(k^2-y^2)]^{1/2} \frac{g(y)}{y-x} dy + C_1 x + C_2 \right\} \times \frac{\operatorname{sgn} x}{[(1-x^2)(x^2-k^2)]^{1/2}}$$

where $k < |x| < 1$, C_1 and C_2 are arbitrary constants, and

$$(C) \quad g(x) = -\frac{1}{\pi} \int_{-1}^1 \left[\frac{1-y^2}{1-x^2} \right]^{1/2} \frac{f(y)}{y-x} dy.$$

For $k=0$, $g(x)$ reduces to H. Söngén's solution [Math. Z. 45, 245-264 (1939)] to the original Prandtl equation.

J. W. Miles (Los Angeles, Calif.).

Parodi, Maurice. Sur une méthode de résolution de certaines équations intégrales. C. R. Acad. Sci. Paris 233, 1253-1254 (1951).

Formal solution of $f(s) = 2s f_0(s^2 + t^2)^{-1/2} F(t)$ for given non-negative integer ν and given $F(t)$. A. Erdélyi.

Functional Analysis, Ergodic Theory

*Nakano, Hidegorô. Topology and linear topological spaces. Maruzen Co., Ltd., Tokyo, 1951, viii+281 pp.

Ce livre se divise en deux parties. La première (chap. I à V) développe d'une façon classique la théorie générale des espaces topologiques, des espaces uniformes et des espaces métriques. Quelques résultats sur le prolongement à un espace uniforme d'une fonction uniformément continue définie dans une partie fermée de l'espace, semblent nouveaux.

La seconde partie (chap. VI à XII) est consacrée aux espaces vectoriels topologiques sur le corps des nombres réels. Après un chapitre d'algèbre linéaire, la notion d'espace vectoriel topologique est introduite au moyen des systèmes de voisinages de 0 (dans un Appendice, l'auteur montre l'équivalence de cette définition avec celle de Kolmogoroff, qui postule la continuité de $x+y$ et λx). Les notions d'ensemble borné et d'espace bornologique ("standard linear topology" dans la terminologie de l'auteur) sont introduites dès le chap. VIII, ainsi que la notion de topologie faible. Le chap. IX est consacré à la notion d'espace dual d'un espace vectoriel topologique E , qui n'est pas défini par l'auteur comme l'espace des formes linéaires continues sur E , mais bien comme l'espace des formes linéaires bornées sur E ; les seules topologies considérées sur ce "dual" ("adjoint space" dans la terminologie de l'auteur) sont la topologie faible et la topologie de la convergence uniforme dans les parties bornées de E . La notion d'espace tonnelé (sous le nom de "normal linear topology") est ensuite introduite, et le théorème de Banach-Steinhaus démontré pour les espaces tonnelés et bornologiques (cas où la notion de "dual" de l'auteur coïncide avec la notion usuelle). Le critère de complète réflexivité donné par l'auteur est naturellement différent du critère usuel (et visiblement moins simple) vu sa définition différente du dual. Le chapitre se termine par l'étude de la dualité pour les espaces vectoriels métrisables, et pour la topologie localement convexe la plus fine. Au chap. X sont traités les espaces normés et leurs propriétés particulières; les théorèmes de Šmulian et d'Eberlein n'y sont donnés que sous une forme affaiblie. A côté de la notion

classique d'espace uniformément convexe, l'auteur introduit la notion "duale" d'espace "uniformément uni" ("uniformly even"), qui donne aussi une condition suffisante de réflexivité.

Le chap. XI est consacré tout entier à une théorie due à l'auteur, celle des "espaces modulés" ("modulated spaces"). Un "module" sur un espace vectoriel E est une fonction numérique $m(x)$ à valeurs ≥ 0 (finies ou non), symétrique, convexe, croissante et continue à gauche et non identiquement nulle sur toute demi-droite issue de 0, et finie dans un intervalle ouvert contenant 0 dans une telle demi-droite. Les ensembles de la forme $m(x) \leq \lambda$ définissent aussitôt une topologie sur E qui peut être définie par une seule semi-norme; toutefois l'auteur croit utile de développer à nouveau pour ces espaces toutes les notions correspondantes à celles de la théorie des espaces normés, mais où le "module" remplace la norme. L'unique application qu'il donne de cette théorie est une généralisation des espaces L^p , savoir les espaces de fonctions mesurables $x(t)$ pour lesquelles il existe $\lambda > 0$ tel que $\int_0^1 |\lambda x(t)|^{p(t)/p(t)} dt < +\infty$, $p(t)$ étant une fonction mesurable dans $0 \leq t \leq 1$, à valeurs dans l'intervalle $1 \leq p \leq +\infty$.

Enfin le dernier chapitre traite des produits cartésiens et produits tensoriels d'espaces vectoriels topologiques. L'auteur introduit ce dernier (au sens algébrique) comme sous-espace du dual algébrique de l'espace des fonctions bilinéaires; il y définit une famille de "cross-topologies", qui correspondent aux "cross-norms" de Schatten dans le cas d'un produit tensoriel d'espaces normés, et montre qu'il y a deux topologies extrêmes dans cet ensemble de topologies; il raccorde ensuite sa théorie avec celle des "cross-norms"; cette partie est sans doute la plus originale et la plus intéressante du livre.

D'une manière générale, l'auteur ne semble pas avoir fait un effort critique suffisant pour dégager les notions vraiment utiles de celles dont l'intérêt est douteux (telles que la notion de "quasi-norme" ou celle de "module"). Il ne paraît pas non plus s'être aucunement préoccupé d'introduire des notations et une terminologie cohérentes et raisonnables; par exemple, il persiste à employer les vieilles notations $+$, \cdot , Σ , Π pour la réunion et l'intersection, ce qui, dans les espaces vectoriels l'amène à noter $A \times B$ l'ensemble des $x+y$ pour $x \in A$ et $y \in B$! Dans un espace uniforme E , $V \times W$ désigne d'ailleurs le composé $V \circ W$ de deux entourages, et $A \times V$ l'ensemble $V(A)$ pour $A \subseteq E$ et $V \subseteq E$. Une "topologie linéaire" n'est pas du tout une topologie, comme on pourrait le croire, mais un système fondamental de voisinages de 0 pour une telle topologie; une "topologie normale" et une "topologie linéaire normale" n'ont absolument aucun rapport; etc., etc.

Enfin, la bibliographie, qui s'étend pourtant jusqu'à 1950, est très incomplète; les travaux de N. Bourbaki, de W. Eberlein, de L. Schwartz et du rapporteur n'y sont absolument pas mentionnés.

J. Dieudonné (Nancy).

Nikodým, Otton Martin. Sur la clôture faible des ensembles convexes dans l'espace réel linéaire où aucune topologie n'est admise. C. R. Acad. Sci. Paris 234, 1727-1728 (1952).

Nikodým, Otton Martin. Sur les clôtures faible et forte des ensembles convexes dans les espaces linéaires réels abstraits. C. R. Acad. Sci. Paris 234, 1831-1833 (1952).

Let L be a real vector space, $\dim L$ its Hamel dimension. For $E \subseteq L$, let $\operatorname{lin}^1 E$ be the set of all points λx such that for some $y \in E$, E contains the set $\{\lambda x + (1-\lambda)y \mid 0 < \lambda < 1\}$. For ordinal α , let $\operatorname{lin}^\alpha E = \operatorname{lin}^1 \operatorname{lin}^{\alpha-1} E$ if $\alpha-1$ exists and $\operatorname{lin}^\alpha E = \bigcup_{1 \leq \beta < \alpha} \operatorname{lin}^\beta E$ if α is a limit ordinal. The reviewer

[Duke Math. J. 18, 443-466 (1951); these Rev. 13, 354] proved that $\dim L < \aleph_0$ if and only if $\text{lin}^\beta C = \text{lin}^1 C$ for each convex $C \subset L$. The author studies further the iteration of lin , obtaining the following results (C representing a convex subset of L). (1) If $\dim L$ is arbitrary [$\dim L \leq \aleph_0$], then for each C there is an $\alpha \leq \Omega[\alpha < \Omega]$ such that $\text{lin}^\beta C = \text{lin}^\alpha C$ for all $\beta \geq \alpha$. (2) If $\dim L \geq \aleph_0$ [$\dim L > \aleph_0$], then for each $\alpha < \Omega[\alpha \leq \Omega]$ there is a C such that $\text{lin}^\beta C \neq \text{lin}^\alpha C$ for all $\beta < \alpha$.

V. L. Klee, Jr. (Princeton, N. J.).

Sunouchi, Gen-ichirô. Harmonic analysis and Wiener integrals. Tôhoku Math. J. (2) 3, 187-196 (1951).

In this paper the author undertakes to give new and shorter proofs for the Cameron-Martin translation and linear transformation theorems for Wiener integrals [Ann. of Math. (2) 45, 386-396 (1944); Trans. Amer. Math. Soc., 58, 184-219 (1945); these Rev. 6, 5; 7, 127] and of Maruyama's generalization of the translation theorem [Kôdai Math. Sem. Rep. 1950, 41-44; these Rev. 12, 343]. Since the Maruyama theorem has also been proved by Cameron and Graves [Trans. Amer. Math. Soc. 70, 160-176 (1951); these Rev. 12, 718], the author's proof is the third independent proof of this theorem.

There are numerous misprints in the paper, and the argument is so condensed that at times it is difficult to fill in the details rigorously. The justification of equation (3.14) and of line (5) page 195 is particularly difficult. The latter can be justified on the basis of an unpublished thesis of H. Colson [An existence theorem for the generalized Riemann-Stieltjes integral, University of Minnesota, December, 1949], but does not follow easily from well known theorems. As Colson points out, the almost everywhere existence of $\int_0^1 f(t)x(t)dx(t)$ for $f(t) \in L_2[0, 1]$ does not follow directly from that of $\int_0^1 f(t)dx(t)$, and special methods have to be used. It is possible that an extension of Colson's methods could be made to justify (3.14).

R. H. Cameron.

Kaplansky, Irving. Orthogonal similarity in infinite-dimensional spaces. Proc. Amer. Math. Soc. 3, 16-25 (1952).

Soit V un espace vectoriel admettant une base dénombrable, sur un corps F algébriquement clos et de caractéristique $\neq 2$. On suppose donnée sur V une forme bilinéaire symétrique (x, y) non dégénérée, et une transformation linéaire T de V dans lui-même, autoadjointe pour (x, y) , c'est-à-dire telle que $(xT, y) = (x, yT)$. L'auteur aborde le problème de la classification de ces transformations par rapport au groupe orthogonal (groupe laissant invariante la forme (x, y)); alors que dans le cas où V est de dimension finie (les autres hypothèses étant conservées), les diviseurs élémentaires de T sont ses seuls invariants relativement à cette classification, il n'en est plus de même lorsque V est de dimension infinie. L'auteur se restreint aux transformations T localement algébriques (c'est-à-dire que tout vecteur $x \in V$ est annulé par un polynôme $f(T)$ en T , $f \neq 0$); il montre dans ce cas qu'on peut se ramener à n'étudier que les transformations T localement nilpotentes. En fait, il aborde seulement l'étude des T nilpotentes; grâce à la théorie de Prüfer [Math. Z. 17, 35-61 (1923)] des modules sur les anneaux principaux, on peut encore parler ici de diviseurs élémentaires, et l'auteur donne un cas où ces diviseurs élémentaires sont encore les seuls invariants de T pour le groupe orthogonal. Il fait enfin l'étude détaillée du cas $T^2 = 0$: soit alors R l'image de V par T , $N \supset R$ le noyau de T , N' le sous-espace orthogonal à N ; on a $N \supset N' \supset R$ et le théorème principal du mémoire est que les dimensions de R , N'/R et

N/N' forment un système complet d'invariants de T pour le groupe orthogonal.

J. Dieudonné (Nancy).

Umegaki, Hisaharu. Weak topology and compact open topology. Proc. Japan Acad. 27, 177-178 (1951).

Le rapporteur a récemment remarqué qu'un résultat de Banach peut s'interpréter en disant que sur le dual E^* d'un espace de Banach E , la topologie la plus fine induisant sur toute boule la même topologie que la topologie faible $\sigma(E^*, E)$ est la topologie de la convergence uniforme dans les parties fortement compactes de E [Proc. Amer. Math. Soc. 1, 54-59 (1950); ces Rev. 12, 524]. L'auteur prétend donner une autre démonstration de ce résultat, mais en réalité il prouve seulement que la topologie de la convergence compacte et la topologie faible induisent la même topologie sur toute boule, c'est-à-dire la moitié facile du théorème ci-dessus; cela résulte aussitôt d'ailleurs d'un théorème général sur les ensembles équicontinus, que l'auteur s'attribue, mais qui est bien connu dans les cas classiques, et est énoncé de façon générale dans N. Bourbaki, Topologie générale, chap. X, §3, prop. 15 [Actualités Sci. Ind., no. 1084 (1949), p. 35; ces Rev. 12, 40].

J. Dieudonné (Nancy).

Sunouchi, Haruo. On integral representations of bilinear functionals. Proc. Japan Acad. 27, 159-161 (1951).

The author considers bilinear bounded functionals $A(f, g)$ on two Banach spaces $(f \in E_1, g \in E_2)$ to the real or complex numbers. He defines the norm of A to be

$$\|A\| = \sup [|A(f, g)|]; \|f\| = 1, \|g\| = 1.$$

For fixed $f, A(f, \cdot) \in E_2^*$ so that $A(f, g)$ defines a bounded linear operator U on E_1 to E_2^* such that $A(f, g) = U(f)g$ and $\|A\| = \|U\|$. He then applies this result to known representation theorems for linear bounded operators to obtain representation theorems for bilinear bounded functionals.

R. S. Phillips (Los Angeles, Calif.).

Calderón, A. P., and Zygmund, A. A note on the interpolation of linear operations. Studia Math. 12, 194-204 (1951).

In an earlier paper [Annals of Mathematics Studies no. 25, pp. 166-188, Princeton Univ. Press, 1950; these Rev. 12, 255] the authors have given a new proof of the Hausdorff-Young theorem and several extensions of M. Riesz's convexity theorem. The purpose of the present paper is to extend the range of values of the exponents for which the theorems proved in the previous paper are valid. Such extensions are particularly useful for linear operations defined on the classes H^r , where r is any positive number.

R. Salem (Cambridge, Mass.).

Smith, Kennan-T. Sur le théorème spectral. C. R. Acad. Sci. Paris 234, 1024-1025 (1952).

The author gives a new proof of the spectral theorem for self-adjoint operators T on a Hilbert space, utilizing the theory of (LF) spaces developed by Dieudonné and Schwartz [Ann. Inst. Fourier Grenoble 1, 61-101 (1950); these Rev. 12, 417]. After obtaining a mapping $f(T)$ of continuous functions on the real line to operators on the Hilbert space, the theory of (LF) spaces is used to extend this mapping to characteristic functions of intervals $(-\infty, t)$ and thus to obtain the resolution of the identity for T .

R. S. Phillips (Los Angeles, Calif.).

Zimmerberg, H. J. On normalizable transformations in Hilbert space. *Acta Math.* 86, 85-88 (1951).

The author extends results of A. C. Zaenen [*Acta Math.* 83, 197-248 (1950); these Rev. 13, 564] on the existence of non-zero eigenvalues of normalisable bounded linear transformations in Hilbert space. The results are applied to systems of linear integral equations, of the kind considered by Zaenen and the author [*Duke Math. J.* 15, 371-388 (1948); these Rev. 11, 37]; the theorems on integral equations have been derived independently by W. T. Reid [*ibid.* 18, 41-56 (1951); these Rev. 13, 564]. *G. E. H. Reuter.*

Visser, C., and Zaenen, A. C. On the eigenvalues of compact linear transformations. *Nederl. Akad. Wetensch. Proc. Ser. A.* 55 = *Indagationes Math.* 14, 71-78 (1952).

The theorems of this note concern the characteristic values (c.v.) and singular values (s.v.) of compact linear transformations A, B, \dots in a unitary space R . Theorem 2: If A, B and $C = AB$ have s.v. $\alpha_1 \geq \alpha_2 \geq \dots, \beta_1 \geq \beta_2 \geq \dots$ and $\gamma_1 \geq \gamma_2 \geq \dots$, then $\gamma_1 \gamma_2 \dots \gamma_k \leq \alpha_1 \alpha_2 \dots \alpha_k \beta_1 \beta_2 \dots \beta_k$ ($k = 1, 2, \dots$). This is used to prove a theorem of S. H. Chang [*Trans. Amer. Math. Soc.* 67, 351-367 (1949), Theorem 4; these Rev. 11, 523], to extend a theorem of Ky Fan [*Proc. Nat. Acad. Sci. U.S.A.* 35, 652-655 (1949), Theorem 3; these Rev. 11, 600] and to deduce inequalities for the s.v. of a product $C = C_1 C_2 \dots C_m$, with some factors C_i compact and the other factors bounded. The reviewer observes that Theorems 1 and 2 of the present paper, and the deduction of Chang's theorem, have already been given by A. Horn [*ibid.* 36, 374-375 (1950); these Rev. 13, 565].

G. E. H. Reuter (Manchester).

Taldykin, A. T. Corrections to the paper, "On linear equations in Hilbert space." *Mat. Sbornik N.S.* 30(72), 463 (1952). (Russian)

See *Mat. Sbornik N.S.* 29(71), 529-550 (1951); these Rev. 13, 564.

Štraus, A. V. On characteristic properties of generalized resolvents. *Doklady Akad. Nauk SSSR (N.S.)* 82, 209-212 (1952). (Russian)

A continuation of an earlier paper [same *Doklady* 78, 217-220 (1951); these Rev. 12, 837]. A family R_λ of linear operators on a Hilbert space \mathfrak{H} , whose domain is all of \mathfrak{H} and which depends on the non-real parameter λ is a generalized resolvent of some symmetric operator if and only if (1) for arbitrary λ_0 with $\Im(\lambda_0) \neq 0$ there exists a subspace \mathfrak{L} of \mathfrak{H} with $\overline{R_{\lambda_0} \mathfrak{L}} = \mathfrak{H}$ such that $(R_\lambda - R_{\lambda_0})f = (\lambda - \lambda_0)R_\lambda R_{\lambda_0} f$ for all non-real λ and all f in \mathfrak{L} , and such that $\|R_\lambda \psi\|^2 \leq (\Im(\lambda))^{-1} \cdot (\Im(R_\lambda \psi, \psi))$ for any λ with $\Im(\lambda) > 0$ and any $\psi \perp [I - (\lambda - \lambda_0)R_\lambda] \mathfrak{L}$, (2) the nullspace of R_λ is (0), (3) $R_\lambda = \overline{R_\lambda}$, (4) R_λ is a regular operator function of λ in each of the half-planes. R_λ is a generalized resolvent of a maximal symmetric operator if and only if (2) and (3) hold and for some fixed non-real λ_0 , $R_\lambda - R_{\lambda_0} = (\lambda - \lambda_0)R_\lambda R_{\lambda_0}$ for all non-real λ . Then R_λ is a generalized resolvent of a self-adjoint operator if these conditions hold and further the null space of $I - (\lambda_0 - \overline{\lambda_0})R_{\lambda_0}$ is (0). *B. Crabtree (Durham, N. H.).*

Rothe, Erich H. A remark on isolated critical points. *Amer. J. Math.* 74, 253-263 (1952).

Let $I(x)$ continuously map an open convex subset of a Hilbert space E into a real line. Differentials $d^p I(x) = I(x)$, $d^1 I(x, h_1)$, $d^2 I(x, h_1, h_2)$, \dots are inductively defined, linear in the sense of Banach in h_1, h_2, \dots . Using the classical representation of a functional linear over E by means of an

inner product (x, y) the gradient g^i of I is defined by setting

$$d^i I(x, h_1, \dots, h_{i-1}, h_i) = (g^i(x, h_1, \dots, h_{i-1}), h_i).$$

Suppose I is strictly non-degenerate of order p at o in that $d^i I$ exists for $i \leq p+2$ and is continuous over some neighborhood U of o , $d^i I, \dots, d^{p-1} I$ vanish when $x=o$, and, using the gradient of index $i=1$,

$$\|\text{grad } d^p I(o, h, \dots, h)\| \geq m > 0, \|h\| = 1.$$

Then Hypothesis H is satisfied, in that there exists a neighborhood U of o such that for all non-null $x \in U \cap \{I(x) = I(o)\}$ the vectors $x-o$ and $\text{grad } I(x)$ are linearly independent. This theorem is motivated by an earlier result of the author [*Math. Nachr.* 4, 12-27 (1951); these Rev. 12, 720] that in euclidean space E^n , Hypothesis H on an I of class C'' implies that the alternating sum of the type numbers of $x=o$ as an isolated critical point of $I(x)$ equals the "vector index" of o . The latter result is related to theorems of the reviewer.

M. Morse (Princeton, N. J.).

Naimark, M. A. On a problem of the theory of rings with involution. *Uspehi Matem. Nauk (N.S.)* 6, no. 6(46), 160-164 (1951). (Russian)

Let \mathfrak{E} denote the algebra of all those operators on a separable Hilbert space \mathfrak{H} of the form $\lambda I + B$, where λ is a scalar, I is the identity operator, and B is completely continuous. The algebra \mathfrak{E} is a C^* -algebra (i.e. a uniformly closed self-adjoint algebra of bounded operators on a Hilbert space). The author has shown [*Uspehi Matem. Nauk (N.S.)* 3, no. 5(27), 52-145 (1948); *Amer. Math. Soc. Translation no. 25*, p. 125, Theorem 2, these Rev. 10, 308; 12, 111] that \mathfrak{E} admits only two non-equivalent irreducible representations (adjoint-preserving homomorphisms of \mathfrak{E} into the algebra of bounded operators on a Hilbert space). These are the one-dimensional representation $\lambda I + B \rightarrow \lambda$ and the identity representation $\lambda I + B \rightarrow \lambda I + B$. It is natural to ask whether or not this property characterizes \mathfrak{E} . The problem is not solved here, but the following result is proved which tends to support an affirmative answer. Let \mathfrak{R} be an irreducible C^* -algebra of bounded operators on a separable Hilbert space \mathfrak{H} and assume that R admits only two non-equivalent irreducible representations one of which is one-dimensional and the other is countably-infinite-dimensional. Then, to every maximal commutative subalgebra \mathfrak{A} in \mathfrak{R} , there corresponds a complete orthonormal system in \mathfrak{H} with respect to which each element of \mathfrak{A} is diagonal. Also, the spectrum of every Hermitian operator in \mathfrak{R} is either a finite or countably infinite set each point of which, with the possible exception of one, is an eigen-value.

C. E. Rickart (New Haven, Conn.).

Šreider, Yu. A. On an example of a generalized character. *Mat. Sbornik N.S.* 29(71), 419-426 (1951). (Russian)

The present example is a generalized character $\chi_s(t)$ [which were introduced by the author in *Mat. Sbornik N.S.* 27(69), 297-318 (1950); these Rev. 12, 420] such that for a certain well known singular measure σ associated with the Cantor set $\chi_s(t)$ is almost everywhere (relative to σ) equal to a certain constant γ , where $0 < |\gamma| < 1$. From the mutual correspondence between these generalized characters and the homomorphisms into the complex numbers of the ring of regular measures on the reals (with convolution as multiplication) the following corollary is deduced. If $\sigma(k, s)$ is the k -fold convolution of σ with itself, translated by s (k an integer, s real), then for $k \neq m$ and for arbitrary s and t , $\sigma(k, s)$ and $\sigma(m, t)$ are singular with respect to each other.

I. E. Segal (Chicago, Ill.).

Orihara, Masao. Rings of operators and their traces. Mem. Fac. Sci. Kyūsyū Univ. A. 5, 107-138 (1950).

The author attempts a treatment of numerical traces (=unitarily invariant positive linear functionals) on general weakly closed self-adjoint rings of bounded operators on Hilbert spaces, from the viewpoint initiated by Murray and von Neumann. Unfortunately a number of the proofs and statements are unconvincing, and in particular Lemma 2.2, on which the proof of the existence of a non-trivial trace in a finite factor is based, appears to be unsound.

I. E. Segal (Chicago, Ill.).

Fukamiya, Masanori. On B^* -algebras. Proc. Japan Acad. 27, 321-327 (1951).

The author treats the relation between B^* - and C^* -algebras and obtains several conditions on a B^* -algebra which imply that it is a C^* -algebra. Among other results it is shown that a given B^* -algebra (containing a unit) is C^* if and only if the equations $f(x^*x)=0$ for all f in the collection P of all positive linear functionals on the algebra imply that $x=0$, and that this in turn is valid if and only if $\|x\| = \sup_{f \in P_0} \{f(x^*x)\}^{1/2}$, where P_0 is the subset of P consisting of functionals that are one on the unit. The proof of Theorem 5 was obscure to the reviewer. I. E. Segal.

Pallu de La Barrière, Robert. Isomorphisme des $*$ -algèbres faiblement fermées d'opérateurs. C. R. Acad. Sci. Paris 234, 795-797 (1952).

Let M be a weakly closed operator ring which has no component of type III. For the case in which M and M' are both of finite type Kaplansky [same C. R. 231, 485-486 (1950); these Rev. 12, 186] has generalized the von Neumann-Murray numerical invariant C , defined for factors, to a function $x \rightarrow C(x)$ whose domain is the set of maximal ideals in the center of M . In this note the author indicates how Kaplansky's definition may be extended so as to be effective even when M and M' are not of finite type and states a number of theorems about the resulting invariant. For example: (a) M is always algebraically isomorphic to an M' for which C is identically one. (b) Every M for which C is identically one is obtainable in a certain specified fashion from what the author has called an Ambrose space [ibid. 233, 997-999 (1951); these Rev. 13, 473]. (c) If M is of finite type or if both M and M' are uniformly infinite then a necessary and sufficient condition that an algebraic isomorphism of M with a second ring M_1 be spatial is that the invariants C and C_1 of M and M_1 be identical. (d) Every algebraic isomorphism of M with M_1 is continuous in the ultrastrong topology. G. W. Mackey.

Nakamura, Masahiro. The two-sided representations of an operator algebra. Proc. Japan Acad. 27, 172-176 (1951).

The author treats the connection between 2-sided representations of and traces (=central positive linear functionals) on uniformly closed self-adjoint algebras of bounded operators on Hilbert spaces, this being analogous to the known connection between 1-sided (ordinary) representations of and positive linear functionals on such algebras. Among other results it is indicated that any trace induces a 2-sided representation which is irreducible if and only if the given trace is an extreme point of the set of all traces. The proofs are in part incomplete. I. E. Segal.

Umegaki, Hisaharu. On some representation theorems in an operator algebra. I. Proc. Japan Acad. 27, 328-333 (1951).

Using results and terminology of F. I. Mautner [Ann. of Math. (2) 51, 1-25 (1950); these Rev. 11, 324] and I. E. Segal [Bull. Amer. Math. Soc. 53, 73-88 (1947); these Rev. 8, 520], the author proves that every state of a separable C^* -algebra is a direct integral of pure states, and that a normal two-sided representation (in the sense of M. Nakamura [see the preceding review] of a separable C^* -algebra is a direct integral of irreducible two-sided representations. J. M. Cook (Baltimore, Md.).

Barbašin, E. A. The method of sections in the theory of dynamical systems. Mat. Sbornik N.S. 29(71), 233-280 (1951). (Russian)

Let M denote a locally compact, locally euclidean n -dimensional manifold of differentiability-class C^r and let

$$(1) \quad dx_j/dt = X_j(x_1, \dots, x_n), \quad j=1, \dots, n,$$

be a system of equations defined at every point of M , the functions X_j being components of a contravariant vector field of class C^r ($r \geq 1$). The dynamical system (1) determines a family of time-parameterized trajectories, one through each point p of M . Let the function $f(p, t)$ denote the point of M to which p "moves" in time t . A continuous numerical function $F(p)$ is called admissible if it is never zero and if the integrals $\int_0^\infty F(f(p, t))dt$ and $\int_0^{-\infty} F(f(p, t))dt$ are divergent. The existence of an admissible function makes it possible to introduce a new time-variable (if necessary) so that the new time ranges from $-\infty$ to $+\infty$.

The family of trajectories is called linearizable (выпрямляемость=linearization) if it can be mapped isomorphically upon a family of parallel straight lines. The principal theorem of the first part of the paper is as follows: A necessary and sufficient condition that the trajectories of a dynamical system be linearizable is that there shall exist a single-valued function u of class C^r whose time-derivative, $u' = \sum_{j=1}^n X_j \partial u / \partial x_j$, is an admissible function. The example in the plane: $dx/dt = \cos y$, $dy/dt = -\sin y$ with the everywhere positive function $u = e^y \cos y$ is offered to show that the condition of "admissibility" cannot be materially weakened.

Following Niemytzki, the system (1) is called completely unstable if to each point p there exists a neighborhood $U(p)$ and a time $t_0 > 0$, such that $f(U, t) \cap U$ is empty for all $t > t_0$. Theorem 1.8: The system (1) is completely unstable if there exists on M a single-valued function u which has a continuous everywhere positive time-derivative.

§2 is on the existence of integrals of an equation in partial derivatives of the first order: $\sum X_j \partial u / \partial x_j = \varphi$, where the X_j and φ are functions of (x_1, \dots, x_n) of class C^r defined in a domain G of a euclidean E_n . It is supposed that a system (1) is defined on G , and that G is filled up by arcs of trajectories of (1). The system of arcs is called unstable if every point p of G moving on its trajectory exists from every compact subset of G as $t \rightarrow \infty$ or $t \rightarrow -\infty$. Lemma 2.1 reads: If the system of arcs filling G is unstable and has no improper saddle (in the sense of Niemytzki) then the domain admits a section. Lemma 2.4 says, that if moreover the time-length of the arcs of G is bounded away from zero, then there is a section S of class C^r . A principal theorem of this section is Theorem 2.7: If G is homeomorphic to a three-dimensional euclidean space and is filled with arcs of the system: $dx_i/dt = X_i(x_1, x_2, x_3)$, satisfying the conditions of Lemma 2.4, then there exist two first integrals u_1 and u_2 of the sys-

tem, such that the rank of the matrix $\|\partial u_i/\partial x_k\|$ is equal to 2 everywhere in G .

Section 3 is on linearizability and stability in the sense of Liapounoff. The theorems are too long to quote. A final consequence, asserted to strengthen a result of Massera [Ann. of Math. (2) 50, 705-721 (1949); these Rev. 11, 721], reads: In the domain of influence of an asymptotically stable point there exists a positively defined function of Liapounoff of class C^r , which has a negatively defined time-derivative. If the phase-space of a dynamical system is a euclidean space, the system (1) is called uniformly dispersed provided there exists a number $R>0$ such that for every $N>R$ there exists a $t_0>0$ with the property that the relation $f(I(R), t) \supset I(N)$ holds for all $t>t_0$ (where $I(R)$ is a full-sphere of radius R centered on the origin). A function $v(x_1, \dots, x_n)$ defined outside of some sphere is called infinitely-big if for every $A>0$ there exists a number R such that $v>A$ outside of $I(R)$. Theorem 4.1: In order that a dynamical system should be uniformly dispersed it is necessary and sufficient that there should exist an infinitely-big function v of class C^r which has a time-derivative everywhere positive in its domain of definition. Section 5 is concerned with the insensitivity (grubost') of the properties studied to small changes in defining conditions. Thus: Theorem 5.1: If the system (1) is linearizable, there exists an everywhere positive function $\eta(x_1, \dots, x_n)$ such that the system $dx_j/dt = X_j + \varphi_j$, $j=1, \dots, n$, is also linearizable, provided the functions φ_j satisfy $\sum \eta^2 \varphi_j^2 < \eta^2$.

Part 2 is devoted to the problem of finding a sectioning surface of the trajectories of a dynamical system defined on a compact orientable differentiable manifold of class C^r ($r \geq 4$). There is some discussion of a theorem by G. Birkhoff [reference is to a Russian translation of "Dynamical Systems" (1941), p. 129] on the existence of an "angular coordinate". The author uses the theory of differential forms of Cartan. His principal results are the following. Theorem 1.1: If a dynamical system of class C^r has a closed admissible differential form of class C^{r-1} , then it has a sectioning surface of class C^r . Theorem 1.2: If a dynamical system of class C^r has a sectioning surface (even if this surface is not smooth) then it has a closed admissible differential form of class C^{r-1} . There are two pages of examples, followed by a discussion of the "sensitivity" of the property of being an admissible closed differential form, and of the property of having a sectioning surface. There is a final section on integral invariants. L. Zippin (Brooklyn, N. Y.).

Ryll-Nardzewski, C. On the ergodic theorems. I. Generalized ergodic theorems. Studia Math. 12, 65-73 (1951).

Let \mathcal{E} be a σ -field of subsets E of a space X , and let μ be a countably additive measure defined on \mathcal{E} with $\mu(X) < \infty$. Let φ be a mapping of X into itself such that (i) $E \in \mathcal{E}$ implies $\varphi^{-1}(E) \in \mathcal{E}$, (ii) $\mu(E) = 0$ implies $\mu(\varphi^{-1}(E)) = 0$, but not necessarily measure preserving ($\mu(E) = \mu(\varphi^{-1}(E))$). Let $L(\mu)$ be the L^1 -space on X with respect to \mathcal{E} and μ . Consider the following conditions: (B) for each $f \in L(\mu)$ there exists a $g \in L(\mu)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\varphi^k(x)) = g(x)$$

μ -almost everywhere on X ; (N) for each $f \in L(\mu)$ there exists a $g \in L(\mu)$ such that

$$\lim_{n \rightarrow \infty} \int_X \left| \frac{1}{n} \sum_{k=0}^{n-1} f(\varphi^k(X)) - g(x) \right| \mu(dx) = 0;$$

(DM) there exists a constant K such that

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu(\varphi^{-k}(E)) \leq K \mu(E)$$

for all $E \in \mathcal{E}$ and for $n=1, 2, \dots$; (H) there exists a constant K such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(\varphi^{-k}(E)) \leq K \mu(E)$$

for all $E \in \mathcal{E}$.

It is clear that (i) (DM) implies (H). The author proves that (ii) (B) and (H) are equivalent, (iii) (N) and (DM) are equivalent, and (iv) (H) does not necessarily imply (DM). (ii) is the main result of this paper. (In case μ is σ -finite, (H) is replaced by other conditions, for example, by (H₂) there exists a constant K such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(Y \cap \varphi^{-k}(E)) \leq K \mu(E)$$

for all $E \in \mathcal{E}$ and for all $Y \in \mathcal{E}$ with $\mu(Y) < \infty$.) The proof of (ii) is based on the idea of Y. N. Dowker [Duke Math. J. 14, 1051-1061 (1947); these Rev. 9, 359] of using the Banach limit. (iii) was originally obtained by Dunford and Miller [Trans. Amer. Math. Soc. 60, 538-549 (1946); these Rev. 8, 280] who also proved that: (v) (N) implies (B). It is to be observed that (v) is now an immediate consequence of (i), (ii) and (iii). Finally, (iv) is proved by constructing a counter-example which is a modification of that of Y. N. Dowker [Bull. Amer. Math. Soc. 55, 379-383 (1949); these Rev. 10, 718]. S. Kakutani (New Haven, Conn.).

Ryll-Nardzewski, C. On the ergodic theorems. II. Ergodic theory of continued fractions. Studia Math. 12, 74-79 (1951).

Let

$$x = \frac{1}{|c_1(x)|} + \frac{1}{|c_2(x)|} + \frac{1}{|c_3(x)|} + \dots$$

be a continued fraction expansion of an irrational number x ($0 < x < 1$), and put

$$\delta(x) = \frac{1}{|c_1(x)|} + \frac{1}{|c_3(x)|} + \dots = \frac{1}{x} - \left[\frac{1}{x} \right],$$

where $c_n(x)$ are positive integers and $[\alpha]$ denotes the integral part of α . As a mapping of the set X of all irrational numbers x with $0 < x < 1$ onto itself, δ is not one-to-one (in fact, δ is an infinity-to-one mapping), but is measure preserving with respect to the measure $\nu(E)$ defined for all Lebesgue measurable subsets E of X by $\nu(E) = (\log 2)^{-1} \int_E (1+x)^{-1} dx$, i.e. $\nu(\delta^{-1}(E)) = \nu(E)$ for any Lebesgue measurable subset E of X . The measure $\nu(E)$ satisfies

$$m(E)/2 \log 2 \leq \nu(E) \leq m(E)/\log 2$$

for all E , where $m(E)$ is the Lebesgue measure of E with the normalization $m(X) = 1$. (From this follows that ν -integrability is equivalent to Lebesgue integrability.)

The author first shows that δ is indecomposable, i.e. that any Lebesgue measurable subset E of X with $\delta^{-1}(E) = E$ satisfies either $\nu(E) = 0$ or $\nu(E) = 1$. Essentially the same result was previously obtained by K. Knopp [Math. Ann. 95, 409-426 (1926)]. From Birkhoff's individual ergodic theorem then follows that for any Lebesgue integrable function $f(x)$ defined on X ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\delta^k(x)) = \frac{1}{\log 2} \int_0^1 \frac{f(x)}{1+x} dx$$

for almost all x . The author applies this result to the function $f(x) = \log c_1(x)$ and obtains the result of A. Khintchine [Compositio Math. 1, 359-382 (1935)]:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\prod_{k=1}^n c_k(x)} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n(n+2)}\right)^{\frac{\log n}{\log 2}}$$

for almost all x . If $f(x)$ is the characteristic function of the set of all $x \in X$ such that $c_1(x) = p$, then we have the result of P. Lévy [ibid. 3, 286-303 (1936)]: for any positive integer p and for almost all $x \in X$, the frequency of p in the sequence $\{c_n(x) | n=1, 2, \dots\}$ exists and is equal to $(\log 2)^{-1} \log((p+1)^2/p(p+2))$.

S. Kakutani (New Haven, Conn.).

Hartman, S. Quelques propriétés ergodiques des fractions continues. Studia Math. 12, 271-278 (1951).

The author follows the idea of C. Ryll-Nardzewski [see the preceding review; we use the same notation as in the preceding review], and applies Birkhoff's individual ergodic theorem to the case when $f(x)$ is non-negative and non-integrable. In this case, from the indecomposability of δ follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\delta^k(x)) = +\infty$$

for almost all x . If $f(x) = c_1(x) = [1/x]$, then we obtain the result of A. Khintchine [Compositio Math. 1, 359-382 (1935)]:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n c_k(x) = +\infty$$

for almost all x . If $f(x) = c_2(x)/c_1(x)$ or $f(x) = c_1(x)/c_2(x)$, then $f(x)$ is non-integrable and hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{c_{k+1}(x)}{c_k(x)} = +\infty, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{c_k(x)}{c_{k+1}(x)} = +\infty$$

for almost all x . Further, if $\{\gamma_n\}$ is a sequence of positive integers with $\gamma_n \rightarrow \infty$ then, for almost all x , the frequency of integers n for which $c_n(x) \geq \gamma_n$ exists and is equal to 0.

S. Kakutani (New Haven, Conn.).

Hartman, S., Marczewski, E., et Ryll-Nardzewski, C. Théorèmes ergodiques et leurs applications. Colloquium Math. 2, 109-123 (1951).

Exposition of recent results in ergodic theory centering around Birkhoff's individual ergodic theorem and its applications. The mapping φ in question is measure preserving, but not necessarily one-to-one. This makes it possible to apply the individual ergodic theorem to the mapping $\varphi(x) = 2x \pmod{1}$ or to the shift transformation of a "one-sided" infinite direct product measure space, and to obtain the classical result of Borel concerning normal numbers or the strong law of large numbers of Kolmogoroff. Applications of the individual ergodic theorem to the problems of continued fractions are also discussed. Most of the results discussed in this paper can be found in a paper by F. Riesz [Comment. Math. Helv. 17, 221-239 (1945); these Rev. 7, 255] and three papers by C. Ryll-Nardzewski and S. Hartman [see the reviews of these three papers above].

S. Kakutani (New Haven, Conn.).

Calculus of Variations

Magenes, Enrico. Sul minimo semi-forte degli integrali di Fubini-Tonelli. Rend. Sem. Mat. Univ. Padova 20, 401-424 (1951).

By an indirect proof, the author shows that the following conditions are sufficient for a pair of functions $y_1(x)$, $a \leq x \leq b$, $y_2(x)$, $c \leq x \leq d$, to furnish a minimum for the integral

$$I(y_1, y_2) = \int_a^b \int_c^d f(x, z, y_1(x), y_2(z), y_1'(x), y_2'(z)) dz dx$$

in the class of functions $y_1(x)$, $y_2(z)$, having the same end-values, lying sufficiently near $y_1(x)$, $y_2(z)$, and having $|y_1'(x) - \bar{y}_1'(x)| \leq N$, $|y_2'(z) - \bar{y}_2'(z)| \leq N$ almost everywhere, where N is a fixed constant. These conditions are:

- $y_1(x)$, $y_2(z)$ are of class C' and satisfy the Euler integro-differential equations;
- $f_{y_1 y_1'}(x, z, \bar{y}_1(x), \bar{y}_2(z), \bar{y}_1'(x), \bar{y}_2'(z)) > 0$ for $a \leq x \leq b$, $c \leq z \leq d$, $|y_1' - \bar{y}_1'| \leq N$, $f_{y_2 y_2'}(x, z, \bar{y}_1(x), \bar{y}_2(z), \bar{y}_1'(x), \bar{y}_2'(z)) > 0$ for $a \leq x \leq b$, $c \leq z \leq d$;
- $\delta_1(x, z, \bar{y}_1(x), \bar{y}_2(z), \bar{y}_1'(x), \bar{y}_2'(z); y_1') > 0$ for $a \leq x \leq b$, $c \leq z \leq d$, $|y_2' - \bar{y}_2'| \leq N$, $0 < |y_1' - \bar{y}_1'| \leq N$, $\delta_2(x, z, \bar{y}_1(x), \bar{y}_2(z), \bar{y}_1'(x), \bar{y}_2'(z); y_2') > 0$ for $a \leq x \leq b$, $c \leq z \leq d$, $0 < |y_2' - \bar{y}_2'| \leq N$;
- the second variation is positive.

L. M. Graves (Chicago, Ill.).

Kimball, William Scribner. Sur le signe des conditions de Weierstrass et de Legendre pour les minima et maxima en calcul des variations. C. R. Acad. Sci. Paris 234, 1021 (1952).

Remark on the sign of the Weierstrass E -function when the direction of integration is reversed. L. M. Graves.

Fichera, Gaetano. Esistenza del minimo in un classico problema di calcolo delle variazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 11, 34-39 (1951).

The author exhibits a formula for a function $u_0(x)$ minimizing $I(u) = \int_A (\Delta u)^2 dx_1 dx_2 \dots dx_n$, where Δ indicates the Laplace operator, and A is a bounded open region in n -space, with a boundary \bar{A} which is piecewise of class C'' . The functions $u(x)$ admitted are those of class C'' in A , whose first and second partial derivatives are in L^2 on A , and which take, together with their normal derivatives on \bar{A} , prescribed boundary values almost everywhere on \bar{A} (in a suitable limiting sense). The proof depends on the use of Green's formula, and of certain complete orthonormal systems for the region A and for a suitably chosen set B contained in the complement of \bar{A} . It is indicated that a more detailed exposition of the method and of its scope of applicability will be published later.

L. M. Graves.

Sigalov, A. G. Conditions for the existence of a minimum of double integrals in an unbounded region. Doklady Akad. Nauk SSSR (N.S.) 81, 741-744 (1951). (Russian)

In a previous note [same Doklady 73, 891-894 (1950); these Rev. 12, 268], the author studied the existence of a minimum for $\iint F(x, y, z, p, q) dx dy$ when the admissible $z(x, y)$ are defined in a bounded domain D , possess a given boundary function, and are uniformly bounded ($|z| < z_0$). He now discards the assumption $|z| < z_0$, and establishes the existence of a bounded minimizing sequence (so that

the problem reduces to the bounded one). In so doing he weakens a previous assumption $\alpha > 1$ to $\alpha \geq 1$ and introduces an additional condition. Let $g(z, L) = \inf F/|p+iq|$ for x, y, p, q where $(x, y) \in D$ and $|p+iq| \leq L$ and let

$$[h(z)]^{-1} = \inf F(x, y, z, 0, 0)$$

for $(x, y) \in D$. The additional condition, which he supposes satisfied for some $L > 0$, is then: There exists an $f(n)$ such that $\int gh \, ds > n$ whenever this integral extends over a finite sum e of disjoint z -intervals of measure $m(e) > n$ situated in $|z| > f(n)$.

L. C. Young (Madison, Wis.).

Hestenes, Magnus R. Applications of the theory of quadratic forms in Hilbert space to the calculus of variations. *Pacific J. Math.* 1, 525-581 (1951).

This paper treats the theory of the second variation in the calculus of variations from the point of view of Hilbert space. A characterization is given of the class of quadratic forms in Hilbert space which are of interest in the calculus of variations, and interrelations between various properties of such forms are discussed. Focal points and conjugate points of quadratic forms are defined relative to certain one-parameter families of closed linear subspaces of Hilbert space. Comparison theorems expressed in these terms are readily obtained. Various applications in the calculus of variations are indicated.

L. M. Graves (Chicago, Ill.).

✓ **Morrey, Charles B., Jr.** The problem of Plateau on a Riemannian manifold and related topics. *Proceedings of the International Congress of Mathematicians*, Cambridge, Mass., 1950, vol. 2, pp. 180-188. Amer. Math. Soc., Providence, R. I., 1952.

The method used by the author is a generalization of the following variant of the solution of the Plateau problem in the Euclidean case: If E, F, G are the first fundamental quantities relative to a representation of a surface S , then one can replace the area-integral by the integral of $E+G$ for the purpose of solving the problem. The latter integral, as a sum of Dirichlet integrals, is treated then in terms of harmonic functions. In the extension of the Plateau problem to surfaces located in a Riemannian manifold, new difficulties arise in connection with the construction of equicontinuous minimizing sequences, and also in connection with the class of functions which should correspond to harmonic functions in the Euclidean case. The paper is an account of the methods used to avoid or to meet these difficulties, and also of the results obtained by the author in this field.

T. Radó (Columbus, Ohio).

Nyström, E. J. Anschauliches zur Variationsrechnung. *Math.-Phys. Semesterber.* 2, 207-216 (1952).

Theory of Probability

✓ **Lévy, Paul.** Arithmétique et calcul des probabilités. *Congrès International de Philosophie des Sciences*, Paris, 1949. Vol. IV, Calcul des probabilités, pp. 125-133. *Actualités Sci. Ind.*, no. 1146. Hermann & Cie., Paris, 1951. 860 francs.

Expository article reviewing some of the number-theoretic theorems which are suggested by probability arguments. These include the prime number theorem, the twin prime problem and the Riemann hypothesis, the last, if the heuristic approach is valid, even at the second approximation. It is

remarkable that the more elementary results predicted by such arguments have mostly turned out to be true.

K. L. Chung (Ithaca, N. Y.).

Dugué, Daniel. Sur certains exemples de décomposition en arithmétique des lois de probabilité. *Ann. Inst. H. Poincaré* 12, 159-169 (1951).

Characteristic functions (c. f.) and generating functions (g. f.) will be given up to a constant factor. (1) The c. f. $e^{-\sigma^2 t^2/2} \cos t$ can be decomposed in other than the obvious way if $\sigma^2 > 1/4 \log 2$. (2) If $a > 2c > 0$, $b > 0$, and $b^2 - 4ac > 0$, $e^{-\sigma^2 t^2/2} [a \cos 2t + b \cos t - c]$ for some σ is a c. f. which does not contain a normal component but its square does. (3) The c. f. $f(t) = (1-t^2)e^{-t^2/2}$ is indecomposable but $f(at)f(bt)$ with $a^2 + b^2 = 1$ has a maximum normal component $g(t)$ such that $f(t)/g(t)$ is indecomposable. If $f(s)$ is an arbitrary function let E be the set of α such that $[f(s)]^\alpha$ is a c. f. or g. f. (4) For $H(s) = 1 + 2s - s^2 + 3s^3 + 3s^4$ the set E is all integers > 1 . (5) For $e^{H(s)}$ the set E is all real numbers $\geq \frac{1}{2}$. (6) For $P(s)^{1/a}$, where P is a polynomial with positive coefficients, E is all numbers ka with $k = \text{non-negative integer}$. It is not known if $f(s)$ exists for which E is the set $m + na$ with m , n integer and a irrational, or is the union of the intervals $(\frac{1}{2}, \frac{1}{2})$ and $(1, \infty)$.

K. L. Chung (Ithaca, N. Y.).

Dugué, Daniel. Sur les produits de variables aléatoires. *C. R. Acad. Sci. Paris* 233, 1421-1422 (1951).

For a non-negative random variable X with distribution $F(x)$ define the multiplicative characteristic function (m. c. f.) to be $P_X(s) = \int_0^\infty x^s dF(x)$ and its analytic continuation for complex s . The inverse of the Pareto variable has density ax^{-1} ($a > 0$) in $(0, 1)$ and m. c. f. $a/(a+s)$. It is announced that if the m. c. f. of X is a meromorphic function of order less than 2 without zero and with all its poles on the negative real axis, then X is the product of a finite or infinite number of inverses of the Pareto variables. Examples: the exponential, normal, chi square, and incomplete gamma distributions.

K. L. Chung (Ithaca, N. Y.).

Pennanéc'h, F. Interprétation géométrique des probabilités. *Cahiers Rhodaniens* no. 3, 16 pp. (1951).

Let X be a random variable which can assume only finitely many values. The author defines a corresponding Hermitian operator A on a finite-dimensional unitary space, and an element ϕ in the space, such that the expectation of any function $f(X)$ of X is given by the inner product of $f(A)\phi$ with ϕ . This representation, which arises naturally in quantum theory without the stated restrictions on X , is then generalized to pairs of random variables.

J. L. Doob (Urbana, Ill.).

Lindley, D. V. The theory of queues with a single server. *Proc. Cambridge Philos. Soc.* 48, 277-289 (1952).

The problem of a single waiting line before a service gate or counter is formulated in general fashion; times between successive arrivals, and service times, are each supposed to follow a general law of distribution, limited only by having a finite mean. Using results of Feller [*Trans. Amer. Math. Soc.* 67, 98-119 (1949); these *Rev.* 11, 255] and extensions of these by Chung and Fuchs [*Mem. Amer. Math. Soc.*, no. 6 (1951); these *Rev.* 12, 722], the author proves that intuitive expectations for the existence of a waiting time distribution are fulfilled; roughly speaking, the conditions are that the gate should not be overloaded: arrivals appearing faster than they can be served, on the average. For random arrivals, he verifies a result for the characteristic

function of the waiting time distribution previously given by Pollaczek [Math. Z. 32, 64-100, 729-750 (1930)]. He also examines in some detail the distribution of waiting times, when arrivals are at regular intervals, and service times have a gamma-type distribution, a problem which for service times with an exponential distribution has similarity to the one-trunk telephone problem with constant holding time. Erlang's tables [Brockmeyer, Halström, and Jensen, Trans. Danish Acad. Tech. Sci. 1948, no. 2 (1948), pp. 197-198; these Rev. 10, 385] for the roots of a certain transcendental equation, of which the author seems unaware, would have been helpful.

J. Riordan.

Perfect, Hazel. On positive stochastic matrices with real characteristic roots. Proc. Cambridge Philos. Soc. 48, 271-276 (1952).

The author, continuing the investigations of Suleimanova [Doklady Akad. Nauk SSSR (N.S.) 66, 343-345 (1949); these Rev. 11, 4], finds that in order that the real numbers 1, a , b , with $|a|, |b| < 1$, shall be characteristic roots of a positive stochastic matrix of order 3 with three linearly independent characteristic vectors, it is necessary and sufficient that $1+a+b$ be positive. The corresponding condition for fourth order matrices is sufficient, but is shown by a counterexample not to be necessary, contrary to Suleimanova's assertion.

J. L. Doob (Urbana, Ill.).

Gnedenko, B. V., and Rvačeva, E. L. On a problem of comparison of two empirical distributions. Doklady Akad. Nauk SSSR (N.S.) 82, 513-516 (1952). (Russian)

Consider two sets of n independent observations on a random variable with a continuous distribution. Let $F_1(x)$ and $F_2(x)$ be resp. the empirical distributions determined by the first and second set; $D_n^+ = \max_x \{F_1(x) - F_2(x)\}$, $D_n^- = \max_x \{F_2(x) - F_1(x)\}$, $D_n = \max\{D_n^+, D_n^-\}$. Now arrange all $2n$ observations in an increasing sequence $x_1 < x_2 < \dots < x_{2n}$. Let ξ_k be $+1$ or -1 according as x_k belongs to the first or second set; $S_k = \xi_1 + \dots + \xi_k$. Then $nD_n^+ = \sup_{1 \leq k \leq 2n} S_k$, $nD_n^- = -\inf_{1 \leq k \leq 2n} S_k$. By this ingenious device the problems concerning the discrepancy between the two empirical distributions are reduced to classical random walk problems concerning S_k . Note that all the possible permutations of the ξ_k 's, subject to the condition $S_{2n} = 0$, are equally likely so that the problem is to count the number of certain conditioned paths subject always to $S_{2n} = 0$. This is done by the well-known method of reflections (or images). In a previous note Gnedenko and Korolyuk [same Doklady 80, 525-528 (1951); these Rev. 13, 570] applied this method to derive the exact distributions for D_n^+ and D_n . Now the present authors do it for their joint distribution and its limit form. The reviewer remarks that the explicit combinatorial formulas needed for all these cases were given already by Bachelier [Calcul des probabilités, vol. 1, Gauthier-Villars, Paris, 1912, pp. 252-253]. For a recent quotation in easy notations see a paper of the reviewer [Trans. Amer. Math. Soc. 64, 205-233 (1948), pp. 215-216; these Rev. 10, 132].

K. L. Chung.

Gnedenko, B. V. Some results on the maximum discrepancy between two empirical distributions. Doklady Akad. Nauk SSSR (N.S.) 82, 661-663 (1952). (Russian)

In addition to the notations used in the preceding review, let $\alpha = [x\sqrt{(2n)}]$, $\beta = [y\sqrt{(2n)}]$. It is shown (local limit

theorems) that

$$\begin{aligned} \sqrt{(2n)} P\{D_n^+ = \alpha/n\} - 4xe^{-2x^2} &\rightarrow 0; \\ \sqrt{(2n)} P\{D_n = \alpha/n\} - dK(x)/dx &\rightarrow 0, \end{aligned}$$

where $K(x)$ is the Kolmogorov limit distribution

$$\sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2 x^2},$$

and

$$2nP\{D_n^- = \alpha/n, D_n^+ = \beta/n\} - \partial^2 S(x, y)/\partial x \partial y \rightarrow 0,$$

where $S(x, y)$ is the Smirnov (joint) limit distribution. Asymptotic expansions of the distributions for $\sqrt{(1/n)}D_n^+$, $\sqrt{(1/n)}D_n$, and $\sqrt{(1/n)}D_n^-$, $\sqrt{(1/n)}D_n^+$ are given explicitly up to and including the term in $1/n$. All these results follow easily from the exact formulas obtained in the previous papers [see the preceding review and first reference there].

K. L. Chung (Ithaca, N. Y.).

Gnedenko, B. V., and Mihalevič, V. S. On the distribution of the number of excesses of one empirical distribution function over another. Doklady Akad. Nauk SSSR (N.S.) 82, 841-843 (1952). (Russian)

Let $x_1 \leq \dots \leq x_n$ and $y_1 \leq \dots \leq y_n$ be the two sets of observations in the preceding review. Let C_n be the number of x_k , $k=1, \dots, n$ for which $F_1(x_k) \geq F_2(x_k)$. It is proved that $P\{C_n = k\} = 1/(n+1)$ for $k=0, 1, \dots, n$. Using the same device as above the problem is reduced to one of random walk, namely the number of positive terms in $S_1, S_2, \dots, S_{2n-1}$. The reviewer remarks that in this form the theorem is equivalent to one given by Chung and Feller [Proc. Nat. Acad. Sci. U. S. A. 35, 605-608 (1949); these Rev. 11, 444].

K. L. Chung (Ithaca, N. Y.).

Yoshida, Kōsaku. Stochastic processes built from flows. Proc. Japan Acad. 26, no. 8, 1-3 (1950).

Let $\{F_t, -\infty < t < \infty\}$ be a group of measure preserving transformations (flow) of a separable measure space R onto itself, with $F_t F_s = F_{t+s}$, $F_0 = \text{identity}$. If $f \in L_p(R)$, define $(Tf)(x) = f(F_t x)$. It is supposed that $T_t f$ is strongly continuous in the L_p topology. Then each T_t is a transition operator in $L_p(R)$ (the qualification means that if $f \geq 0$ then $T_t f \geq 0$ and the two functions have the same norm), and the semigroup $\{T_t, t \geq 0\}$ has the infinitesimal generator A , with $T_t = e^{tA}$. Moreover A^2 is an infinitesimal transition operator, generating a semigroup $\{e^{tA^2}, 0 \leq t\}$. The Fokker-Planck equation $\partial f(t, x)/\partial t = A^2 f$ is then integrable stochastically if $A = p' \partial / \partial x^i$ (usual summation convention) is the infinitesimal transformation of a one-parameter Lie group of measure preserving transformations. It is shown that if A_1, \dots, A_n correspond to flows, and if (h^i) is symmetric and positive definite, then $h^i A_i$ is the infinitesimal generator of a one parameter semigroup of transition operators. In particular, if the group of motions of R is a semisimple Lie group, the Casimir operator is of the above type, and is the infinitesimal generator of a temporally and spatially homogeneous stochastic operator in R —a generalized Brownian motion in a homogeneous space. (Cf. the following review for an approach to this Brownian motion from a different point of view.)

J. L. Doob (Urbana, Ill.).

Itô, Kiyosi. Brownian motions in a Lie group. Proc. Japan Acad. 26, no. 8, 4-10 (1950).

Let G be an n -dimensional Lie group. A right invariant Brownian motion is a stochastic Markov process (of G -valued random variables) whose sample functions are

almost all continuous, and whose transition probabilities satisfy the usual regularity conditions imposed in studying diffusion processes, besides being stationary in time and right invariant in space. If f is a function of class C_2 , the usual elliptic differential operator D (on f) of the Fokker-Planck equation is then derived, and determines the transition probabilities uniquely. Necessary and sufficient conditions are given that an elliptic operator D with stated regularity properties correspond to a right-invariant Brownian motion. Just as in the classical case, the Brownian motion has independent increments. The corresponding results for left-invariant Brownian motion are also given, as well as conditions necessary and sufficient that the operator D correspond to a bilaterally invariant Brownian motion. The author uses the constructive probabilistic method of stochastic differential equations described in an earlier paper [Nagoya Math. J. 1, 35-47 (1950); these Rev. 12, 425], rather than the semigroup method of Yosida (see the preceding review).

J. L. Doob (Urbana, Ill.).

Sirao, Tunekiti, and Nisida, Tosio. On some asymptotic properties concerning Brownian motion. Nagoya Math. J. 4, 97-101 (1952).

Paul Lévy stated in his book "Processus stochastiques et mouvement brownien" [Gauthier-Villars, Paris, 1948; these Rev. 10, 551] that Kolmogorov and Feller's most precise form of the law of iterated logarithm holds for the Brownian motion process both at infinity and locally. No proof was given because, of course, the result follows easily from the corresponding one for sums of independent random variables. The authors write down such a proof. As a matter of fact I. Petrowsky [Math. Ann. 109, 425-444 (1934)] was the first to prove the theorem.

K. L. Chung.

Egerváry, E., and Turán, P. On a certain point of the kinetic theory of gases. Studia Math. 12, 170-180 (1951).

Let E be a cube whose walls are $x=0, x=\pi, y=0, y=\pi, z=0, z=\pi$. In this cube we assume there are n particles (n large). The particles are assumed to be dimensionless of equal mass, the impacts on the walls follow the laws of elastic impact, and we assume that there are no attractive or repulsive forces between the particles and that there are no exterior forces acting on the particles. Assume further that two particles never collide. At time $t=0$ the coordinates $x_{\nu,0}, y_{\nu,0}, z_{\nu,0}, \nu=1, 2, \dots, n$ are arbitrary and the velocities satisfy

$$(1) \quad \begin{aligned} \dot{x}_{\nu,0} &= (n+\nu)^{1/2}, & \dot{y}_{\nu,0} &= (n+\nu)^{1/2}/\sqrt{2}, \\ \dot{z}_{\nu,0} &= (n+\nu)^{1/2}/\sqrt{3}, & \nu &= 1, 2, \dots, n. \end{aligned}$$

Let K be any parallelepiped of volume V_K contained in E . Denote by $N(t_0, K)$ the number of particles in K at time $t=t_0$. The authors prove that for any $0 \leq t \leq n^{1/4}$, except time intervals whose total length does not exceed $cn^{-1/10}(\log n)^2$ (c an absolute constant) we have for every K

$$\left| \frac{N(t_0, K)}{n} - \frac{V_K}{\pi^3} \right| < n^{-1/10}.$$

The theorem shows that whatever the initial position of the particles if the velocities satisfy (1) "most of the time" there will be equidistribution. The authors obtain the same result for various other assumptions on the velocities, e.g., $\dot{x}_{\nu,0} = (n+\nu)^{1/2}, \dot{y}_{\nu,0} = (n+\nu)^{1/2}/\sqrt{2}, \dot{z}_{\nu,0} = (n+\nu)^{1/2}/\sqrt{3}$. They also obtain the same result if they permit elastic collision between two particles. Also various interesting open questions are discussed.

P. Erdős (Los Angeles).

Norton, Kenneth A., Shultz, Edna L., and Yarbrough, Helen. The probability distribution of the phase of the resultant vector sum of a constant vector plus a Rayleigh distributed vector. J. Appl. Phys. 23, 137-141 (1952).

Presents a table and graphs of the cumulative distribution of the phase of a plane vector expressible as the sum of a constant vector and a vector which is normally distributed with variances equal in all directions. The results are applicable to radio wave propagation.

A. Blake.

Reich, Edgar. The game of "gossip" analyzed by the theory of information. Bull. Math. Biophys. 13, 313-318 (1951).

This paper deals with an analysis of a distributed model of the game of "gossip," in which a message is passed through a line of individuals, and the final (in general, garbled) result is compared with the original ungarbled message. The deterioration of information (defined in the sense of Shannon and Wiener) along the line is calculated, and exact as well as asymptotic formulas suggesting approximate linear electric network analogues are obtained.

Author's summary.

Mandelbrot, Benoit. Sur la notion générale d'information et la durée intrinsèque d'une stratégie. C. R. Acad. Sci. Paris 234, 1345-1347 (1952).

Féron, Robert. Information et corrélation. C. R. Acad. Sci. Paris 234, 1343-1345 (1952).

Pollaczek, Félix. Délais d'attente des avions atterrissant selon leur ordre d'arrivée sur un aéroport à s pistes. C. R. Acad. Sci. Paris 234, 1246-1248 (1952).

Mathematical Statistics

*Barnard, G. A. A theory of mathematical statistics independent of the calculus of probability. Congrès International de Philosophie des Sciences, Paris, 1949. Vol. IV, Calcul des probabilités, pp. 115-124. Actualités Sci. Ind., no. 1146. Hermann & Cie., Paris, 1951. 860 francs.

The author writes: "Thus the distinction between mathematical statistics and the calculus of probability, from this point of view, lies in the fact that in the calculus of probability we are concerned with sets of 'trials', which have special properties of 'independence' and possibilities of combination, which do not exist for members of the populations dealt with in mathematical statistics." His conception of mathematical statistics may be inferred from the following sentence: "A great deal of the theory of mathematical statistics can be expressed in terms of the operators δ , expectation, and \mathfrak{C} , covariance."

J. Wolfowitz.

Rider, Paul R. The distribution of the quotient of ranges in samples from a rectangular population. J. Amer. Statist. Assoc. 46, 502-507 (1951).

The distribution is derived and various properties of it are determined (e.g., the mode and moments). Its use in testing whether two samples are from the same rectangular population is illustrated. Tables of the 1, 5, and 10 per cent points are given for sample sizes $m, n=2(1)10$.

D. F. Votaw, Jr. (New Haven, Conn.).

Banerjee, D. P. On the distribution of the range of variation of the ordered variates in samples of n from normal universe. *Proc. Indian Acad. Sci., Sect. A.* 35, 24-26 (1952).

Formulas are found for the density functions of the range and of the deviation of the mean of the variates from the mean of the extreme values for samples of independent, identically normally distributed variates. In the concluding two formulas of the paper, factors $n^2(n!)^2$ and $n^3(n!)^2$ respectively should be replaced by n^3 . S. W. Nash.

Krull, Wolfgang. Zur Korrelationstheorie zweidimensionaler Merkmale. *Mitteilungsblatt Math. Statist.* 3, 15-29 (1951).

Krull, Wolfgang. Korrelationstheorie mehrdimensionaler Merkmale. *Mitteilungsblatt Math. Statist.* 3, 185-200 (1951).

The first paper deals with 2-variate correlation and the second, with the r -variate case. Let G be the matrix of sums of squares and products, about the means, of a sample from an r -variate population, and H the matrix of residual sums of squares and products after taking out a regression on s determining variables. As measures of the multivariate correlation between the stochastic and determining variables the author proposes: (a) the ratio of the sum of the μ smallest (largest) latent roots of H and the sum of the μ smallest (largest) roots of G for $\mu = 1, \dots, r$; (b) the ratios of the μ th coefficients of the characteristic equations of H and G respectively, especially $\text{tr } H / \text{tr } G$. No sampling theory is discussed. When he wrote the papers, the author was unaware of Hotelling's definition of canonical correlations, viz. the roots of the determinantal equation $|(G-H) - G| = 0$.

A. T. James (Princeton, N. J.).

James, G. S. The comparison of several groups of observations when the ratios of the population variances are unknown. *Biometrika* 38, 324-329 (1951).

Welch, B. L. On the comparison of several mean values: an alternative approach. *Biometrika* 38, 330-336 (1951).

B. L. Welch [*Biometrika* 34, 28-35 (1947)] has described a method for comparing means, or regression coefficients, of two normal populations, when the ratio of their variances is unknown. Here James derives by similar methods a test for the equality of k means or regression coefficients. To test the significance of the differences among x_1, \dots, x_k , he calls for calculating the statistic $\chi^2 = (\sum w x_i^2 - (\sum w x_i)^2 / w)$, which if the ν_i are all large can be taken to have the chi-square distribution with $(k-1)$ degrees of freedom. However, in the contrary case the comparison must be with a new statistic $2h(a)$ of which an expansion in inverse powers of the ν_i (through inverse squares) is given (as a complicated algebraic formula). A further corrective term seems hardly feasible.

Welch examines the procedure of James and proposes an alternative method of computing h (through inverse first powers only) and secures a representation of the latter, by a Pearson type curve. He applies his simplified method to an example, showing numerical agreement with James' method.

A. A. Bennett (Providence, R. I.).

Das, A. C. Systematic sampling. II. *Science and Culture* 15, 441-442 (1950).

This is a continuation of a previous note by the author [*Science and Culture* 15, 157-158 (1949); these Rev. 11, 260].

D. F. Volau, Jr. (New Haven, Conn.).

Neyman, Jerzy. Foundation of the general theory of statistical estimation. *Congrès International de Philosophie des Sciences*, Paris, 1949. Vol. IV, *Calcul des probabilités*, pp. 83-95. *Actualités Sci. Ind.*, no. 1146. Hermann & Cie., Paris, 1951. 860 francs.

Brief exposition of the author's theory of estimation.

J. Wolfowitz (Los Angeles, Calif.).

Bhate, D. H. A note on the estimates of centre of location of symmetrical populations. *Calcutta Statist. Assoc. Bull.* 4, no. 13, 33-35 (1951).

For symmetrical Pearson probability functions the author shows that the mean of two symmetrically placed elements in an ordered sample is more efficient than the median as an estimate of the center of location. An example demonstrates that this statement is not true for all symmetrical probability functions.

L. A. Aroian.

Ogawa, Junjiro. Contributions to the theory of systematic statistics. I. *Osaka Math. J.* 3, 175-213 (1951).

The author considers the general problem of finding optimum point estimates, e.g. best linear unbiased estimates, minimum variance estimates, most efficient estimates, etc., in the class of systematic estimates (i.e. those which are functions of order statistics). Best systematic estimates are derived for the parameters of the normal distribution. The problem of testing hypotheses using systematic statistics is also treated and in several cases the tests are explicitly derived.

R. P. Peterson (Seattle, Wash.).

Levene, Howard. On the power function of tests of randomness based on runs up and down. *Ann. Math. Statistics* 23, 34-56 (1952).

For testing the hypothesis H that n random variables X_1, \dots, X_n are identically and independently distributed the author considers tests based on the signs of the $n-1$ differences $X_{i+1} - X_i$ ($i = 1, \dots, n-1$), and in particular on the number of runs of various lengths among these + and - signs (u -runs) and on related statistics. Extending a result of Wolfowitz [same *Ann.* 15, 163-172 (1944); these Rev. 6, 8] he proves the joint distribution of any fixed set of such u -run statistics to be asymptotically normal under H and also under certain alternatives such as linear and cyclic trends. Following an approach introduced by Wolfowitz [ibid. 13, 247-279 (1942); these Rev. 4, 107] he derives the asymptotic likelihood ratio test for testing H against all continuous alternatives when instead of the complete set of observations only the number of u -runs of various lengths are considered and all runs exceeding a given length are grouped together. Asymptotic power results are obtained by considering certain sequences of alternatives which tend to H in a suitable manner as $n \rightarrow \infty$. These are based on the fact that against such alternatives the asymptotic power depends only on the expectations of the u -run statistics under H and the alternatives, and their covariance matrix under H . The remainder of the paper is devoted to computing these expectations for certain specified parametric alternatives. A table of the exact covariances of u -run statistics is appended.

E. L. Lehmann.

Hemelrijk, J. Note on Wilcoxon's two-sample test when ties are present. *Ann. Math. Statistics* 23, 133-135 (1952).

The parameterfree two-sample test of F. Wilcoxon, depends upon a statistic U defined as the number of pairs (i, j) with $x_i > y_j$, where x_1, \dots, x_n , and y_1, \dots, y_m , are the two

samples. Mann and Whitney [same Ann. 18, 50-60 (1947); these Rev. 9, 151] have derived the probability distribution of U , under the hypothesis that the samples have been drawn independently from the same continuous population. This author notes that Wilcoxon's U is closely connected with the S of M. G. Kendall [Biometrika 30, 81-93 (1938)] in the theory of rank correlation. Indeed $2U + S = nm$. He obtains a formula for cases in which ties may occur and thus prepares for the extension of Wilcoxon's test to samples from any population. A. A. Bennett (Providence, R. I.).

Sevast'yanov, B. A. The theory of branching random processes. Uspehi Matem. Nauk (N.S.) 6, no. 6(46), 47-99 (1951). (Russian)

An elegant expository paper with somewhat simplified proofs of known basic results: functional equation, discrete and continuous cases, asymptotic behavior of branching processes with one and with several types of particles.

M. Loève (Berkeley, Calif.).

Davenport, W. B., Jr., Johnson, R. A., and Middleton, D. Statistical errors in measurements on random time functions. J. Appl. Phys. 23, 377-388 (1952).

Mathematical Biology

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. VIII. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 850-852 (1951).

For a male (female) to belong to a particular race is expressed by the vanishing of n first order differential expressions in $m > n$ attributes, time being the independent variable. The $2n$ equations become side conditions for a problem of Lagrange, the vanishing of the first variation expressing stationarity of the race, and the positivity of the second variation expresses stability. Analogous expressions are written in the case of crossing. A. S. Householder.

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. IX. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 936-937 (1951).

A brief symbolic statement of Mendel's laws.

A. S. Householder (Oak Ridge, Tenn.).

Haimovici, Adolf. On the problem of interaction of two species with one living at the expense of the other. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 213-215 (1949). (Romanian. Russian and French summaries)

L'auteur étudie la coexistence de deux espèces dont la première (x) produit une substance (z), et la deuxième (y) se nourrit de (x) et de (z). L'auteur postule, sans les justifier, les équations intégral-différentielles suivantes

$$(1) \quad \frac{dx}{dt} = x \left[e_1 - \gamma_1 y - \delta_1 z - \int_0^\infty F_1(\tau) y(t-\tau) d\tau \right],$$

$$(2) \quad \frac{dy}{dt} = y \left[-e_2 + \gamma_2 y + \delta_2 z + \int_0^\infty F_2(\tau) x(t-\tau) d\tau \right],$$

$$(3) \quad \frac{dz}{dt} = \alpha_1 x - \alpha_2 y + \int_0^\infty [F_3(\tau) x(t-\tau) - F_4(\tau) y(t-\tau)] d\tau,$$

en supposant que les valeurs de x, y, z dans l'intervalle $-\infty < t < 0$ sont connues et que les fonctions F_i et les paramètres $\alpha_i, e_i, \gamma_i, \delta_i$ sont positifs. Il en tire sans démonstration quelques propositions sur le comportement de cette triple symbiose. Les équations paraissent contenir quelques erreurs dues peut-être à l'imprimerie; par exemple, dans l'équation (2) on devrait avoir $\gamma_2 x$ à la place de $\gamma_2 y$.

V. A. Kostitsin (Paris).

Rapoport, Anatol. "Ignition" phenomena in random nets. Bull. Math. Biophys. 14, 35-44 (1952).

The spread of excitation in a "random net" is investigated. It is shown that if the thresholds of individual neurons in the net are equal to unity, a positive steady state of excitation will be reached equal to γ , which previously had been computed as the weak connectivity of the net. If, however, the individual thresholds are greater than unity, either no positive steady state exists, or two such states depending on the magnitude of the axone density. In the latter case the smaller of the two steady states is unstable and hence resembles an "ignition point" of the net. If the initial stimulation (assumed instantaneous) exceeds the "ignition point," the excitation of the net eventually assumes the greater steady state. (From the author's abstract.)

A. S. Householder (Oak Ridge, Tenn.).

TOPOLOGY

de Bruijn, N. G., and Erdős, P. A colour problem for infinite graphs and a problem in the theory of relations. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 369-373 (1951).

A linear graph is " k -colourable" if each vertex can be given one of k colours in such a way that the end-points of each edge are coloured differently. It is deduced from a theorem of R. Rado [Canadian J. Math. 1, 337-343 (1949); these Rev. 11, 238; see also Gottschalk, Proc. Amer. Math. Soc. 2, 172 (1951); these Rev. 12, 683] that if every finite subgraph of an infinite graph G is k -colourable (where k is finite) then G is k -colourable. Let f be a function assigning to each element s of an arbitrary set S a subset $f(s)$ of S ; $T \subset S$ is "independent" if whenever $a, b \in T$ and $a \neq b$ we have $a \notin f(b)$ and $b \notin f(a)$. The colouring theorem is applied to show that if $f(S)$ always has at most k elements (k finite), S is a union of $2k+1$ independent sets, and that if

$f(S)$ is always finite then S is a union of \aleph_0 independent sets.

A. H. Stone (Manchester).

Cohen, Herman J. Sur un problème de M. Dieudonné. C. R. Acad. Sci. Paris 234, 290-292 (1952).

Consider the following property of a topological space E : (I) The family of all neighbourhoods of the diagonal in $E \times E$ defines a uniform structure on E . Dieudonné [J. Math. Pures Appl. 23, 65-76 (1944); these Rev. 7, 134] has shown that (I) holds if E is paracompact. The author proves (Th. 1) that (I) implies that E is "collectionwise normal" [Bing, Canadian J. Math. 3, 175-186 (1951); these Rev. 13, 264]; an example due to Bing [loc. cit.] thus answers a question raised by Dieudonné by providing a normal space E for which (I) is false. The author also characterizes (I) in terms of open coverings of E , and thence, using another example due to Bing, shows that the converse of Th. 1 is false.

A. H. Stone (Manchester).

Shirota, Taira. On systems of structures of a completely regular space. Osaka Math. J. 2, 131-143 (1950).

For (uniform) structures \mathcal{U} and \mathcal{V} compatible with the topology of a completely regular space R , write $\mathcal{U} \geq \mathcal{V}$ if a function uniformly continuous relative to \mathcal{V} is likewise relative to \mathcal{U} . Let D be the class of all structures, D_i that of all totally bounded ones, D_c that of all complete ones, and D_m that of all which are generated by not more than m relations, m being an infinite cardinal. Typical results are as follows. D (D_m) has a minimum if and only if R is locally compact (and has character not greater than m). $D_c \cap D_m$ has a minimum if and only if R is compact and has character not greater than m . A condition equivalent to $D_i \cap D_m$ having a maximum is established, which reduces in the metric case to R being bicomact. Finally, the cardinal number of D_m , D_c , and D_i is established for general classes of metric spaces.

R. Arens (Los Angeles, Calif.).

Stone, A. H. Incidence relations in multicoherent spaces.

III. Pacific J. Math. 2, 99-126 (1952).

A continuation of the author's earlier studies [Trans. Amer. Math. Soc. 66, 389-406 (1949); Canadian J. Math. 2, 461-480 (1950); these Rev. 11, 45; 12, 349] on relations between systems of sets and their frontiers in a locally connected space of a given degree of multicoherence. Earlier papers dealt with the case of uncoherent spaces and with considerations involving just a pair of sets. Here arbitrary systems of sets are considered and relations between these systems and their frontiers are derived. Among other results a further extension of the Phragmén-Brouwer theorem is made and an addition theorem is established.

G. T. Whyburn (Charlottesville, Va.).

Zarankiewicz, Kazimierz. On the category of the set of cut points of continua of a certain type. Czechoslovak Math. J. 1(76), 57-62 (1951).

It is shown that every n -dimensional continuum C may be embedded in an n -dimensional continuum C^* such that the set K of all cut points of C^* is dense in C^* as is also $C^* - K$. Further, if the set E of all end points of a continuum C is dense in C , $C - E$ is of the first category. Use is made of a definition of the simple link of a continuum C containing a given non-cut point a as the intersection $P(a)$ of all sets A_α for all decompositions of C as the union of two continua A_α and B_α intersecting in a single cut point x of C and where $A_\alpha \supset a$.

G. T. Whyburn (Charlottesville, Va.).
A Russian version of this paper was published in *Czechoslovak Math. J.*

Cairns, Stewart S. An elementary proof of the Jordan-Schoenflies theorem. Proc. Amer. Math. Soc. 2, 860-867 (1951).

This proof has the novel feature that the Schoenflies theorem is derived first, by a complicated but "elementary" approximation construction which uses only easy cases of Jordan's theorem, and Jordan's theorem in full generality is then deduced from it. Some simple properties of convex curves are taken for granted. The diagram on p. 862 is somewhat misleading.

A. H. Stone (Manchester).

Fort, M. K., Jr. Points of continuity of semi-continuous functions. Publ. Math. Debrecen 2, 100-102 (1951).

The author proves that an upper or lower semi-continuous mapping of a topological space into the set of all non-empty compact subsets of a metric space is continuous at points of a residual set.

S. B. Myers (Ann Arbor, Mich.).

Newman, M. H. A. Fixed point and coincidence theorems. J. London Math. Soc. 27, 135-140 (1952).

"Brief review of some of the advances in the theory of fixed points and coincidences of mappings that have occurred during the last twenty-five years."

Extract from the paper.

Glicksberg, I. L. A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points. Proc. Amer. Math. Soc. 3, 170-174 (1952).

Let $x \rightarrow \Phi(x)$ be a point-to-closed convex set mapping of a compact convex subset S of a locally convex Hausdorff linear topological space X into itself such that the graph $\bigcup_{x \in S} (x, \Phi(x))$ is closed in the Cartesian product $X \times X$. Then there exists a fixed point in S , i.e. a point $x_0 \in S$ such that $x_0 \in \Phi(x_0)$. This is a generalization of the results of H. F. Bohnenblust and S. Karlin [Contributions to the theory of games, pp. 155-160, Ann. of Math. Studies, no. 24, Princeton, 1950; these Rev. 12, 844] and the reviewer [Duke Math. J. 7, 457-459 (1941); these Rev. 3, 60]. If Φ is a point-to-point mapping this result is reduced to the fixed point theorem of A. Tychonoff [Math. Ann. 111, 767-776 (1935)], and in this case the author's method of approximating Φ by a finite-dimensional point-to-closed convex set mapping is more natural than the usual approximation by a finite-dimensional point-to-point mapping obtained by simplicial interpolation. The author applies his result to prove the minimax theorem for continuous games with continuous payoff as well as the existence of equilibrium points in the sense of J. F. Nash [Proc. Nat. Acad. Sci. U. S. A. 36, 48-49 (1950); these Rev. 11, 192].

S. Kakutani.

Aleksandrov, P. S. The present status of the theory of dimension. Uspehi Matem. Nauk (N.S.) 6, no. 5(45), 43-68 (1951). (Russian)

This article contains a rather complete survey of the present development of dimension theory; several problems are stated. Typical topics: interrelations of different definitions of the dimension, and the relations between the dimension of a space and properties of its ring of (bounded) continuous functions (§1); various mapping classes which increase (decrease) the dimension (§2); theorem on ϵ -transformation into a polytope and its generalizations (§3); essential mappings (§4); various definitions of the dimension based on the homology theory (§§5, 8); problems concerning the dimension of the topological product of two spaces (§§6, 7). In §§9 and 10, some results are stated concerning new aspects of the homological theory of dimension, which are due to K. Sitnikov.

M. Kaléřov (Prague).

Curtis, M. L., and Young, G. S. A theorem on dimension. Proc. Amer. Math. Soc. 3, 159-161 (1952).

The authors prove that, for a compact metric space X , $\dim X \leq n$ if and only if there exists a light map of X into I^n , and state (in a remark added in proof) that this theorem is closely related to several results contained in a note of the reviewer [Časopis Pěst. Mat. Fys. 75, 1-16 (1950); these Rev. 12, 119]. (It may be noted that the authors' result follows, as a special case, from Theorem 1, loc. cit.)

M. Kaléřov (Prague).

Whyburn, G. T. Open mappings on locally compact spaces. Mem. Amer. Math. Soc., no. 1, i+24 pp. (1950). \$75.

This paper is a contribution to the topologization of complex function theory (problem of Brouwer) and is influenced

by the work of S. Stoilow [Leçons sur les principes topologiques de la théorie des fonctions analytiques, Gauthier-Villars, Paris, 1938] and others. The essential topological properties of an analytic function seem to be that it shall respect open sets and that the inverse of no point shall contain a continuum. The author's results are based on the study of maps $f: X \rightarrow Y$ (with X and Y locally compact, locally connected, connected, separable, metric) which take open sets onto open sets and which are constant on no (non-trivial) connected set. Section 3 is devoted to the proof of a lemma which yields (among others) this result: Let f be a function defined on the region R in R^2 (the complex plane) with values in R^2 and which is continuous and has a non-zero derivative at each point of $R - X$, where X is countable. Then f maps sets open in R onto sets open in R^2 and is not constant on any non-trivial connected set. The results are applied to give elementary proofs of fundamental properties of analytic functions. In section 5 the author considers expansive maps. Here X is the union of a sequence of regions R_1, R_2, \dots satisfying $R_n \subset R_{n+1}$, R_n is compact and any compact $K \subset f(X)$ meets only finitely many sets $f(R_n - R_n)$. Expansive maps are characterized in various ways, examples are given and the general theory is related to concrete results in function theory. As is easily seen, $\exp z$ is not expansive and it is shown that an expansive entire function takes on all values. The remainder of the paper is devoted largely to results of a set-theoretic character involving properties of the map f . Important new concepts are introduced such as quasi-interiority, developability, and approximately interior sequences. These notions are characterized, interrelated and interpreted in function-theoretic terms. Many of the results are generalizations of results of the author [Analytic Topology, Amer. Math. Soc. Colloq. Publ., v. 28, New York, 1942; these Rev. 4, 86] and Stoilow [loc. cit.]. Throughout the paper emphasis is placed on correct, elegant and elementary topological methods as opposed to the heuristic, "geometric" style common to much literature of the subject.

A. D. Wallace (New Orleans, La.).

Dubois-Violette, Mme. Étude des réseaux de courbes tracés sur une surface close et en général localement homéomorphes à un faisceau de droites par parallèles. Ann. Sci. École Norm. Sup. (3) 68, 267-325 (1951).

In this paper a broad study is made of regular curve families filling a closed surface, with the exception of a finite number of singular points. A number of interesting examples are given of families having singularities only of the "col" type (singularities at which a finite number of curves of the family end); in particular, it is shown that such a family can be constructed to contain a preassigned set of closed curves and no other closed curves. A number of results are obtained on the structure of the limit sets of open curves. No references are given to the considerable related literature.

W. Kaplan (Ann Arbor, Mich.).

Moise, Edwin E. Affine structures in 3-manifolds. III. Tubular neighborhoods of linear graphs. Ann. of Math. (2) 55, 203-214 (1952).

Let L be a polyhedral linear graph in E^3 , let γ be a positive number and f a homeomorphic mapping into E^3 of a neighborhood U of L . The main theorem asserts the existence of a simplicial decomposition L_1 of L , a tubular neighborhood T of L_1 in the intersection of U with the γ -neighborhood of L , and a polyhedron K where (1) K is a neighborhood of $f(L)$ in E^3 , (2) there is a homeomorphism of T onto

K mapping each element of the "natural decomposition" of K onto a polyhedron of diameter less than γ and (3) $D(K, f(T)) < 2\gamma$, where $D(K, K')$ is the inf of the numbers ϵ such that there exists a homeomorphism between K and K' with corresponding points everywhere at distance apart less than ϵ . In proving the principal theorem, the author develops a number of lemmas concerning the existence of piecewise linear homeomorphisms satisfying certain conditions, and of extensions of such homeomorphisms. These results and the accompanying concepts should also prove useful in other investigations of topological manifolds. For the first two papers of this series see Ann. of Math. (2) 54, 506-533 (1951); 55, 172-176 (1952); these Rev. 13, 484, 574.

S. S. Cairns (Urbana, Ill.).

Moise, Edwin E. Affine structures in 3-manifolds. IV. Piecewise linear approximations of homeomorphisms. Ann. of Math. (2) 55, 215-222 (1952).

This paper contains the following two theorems, in addition to some lemmas concerning a solid torus in E^3 . Let K be a combinatorial 3-manifold with boundary, where K is also a finite simplicial complex which can be mapped into E^3 by a homeomorphism f . Let ϵ be a positive number. Theorem 1 states that there exists a piecewise linear homeomorphism f' of K into E^3 such that the distance between $f(p)$ and $f'(p)$ is less than ϵ for all p on K . An absolutely knotted 1-sphere in E^3 is characterized by the existence of a self-homeomorphism of E^3 mapping J into a plane. Theorem 2 states that, for any such 1-sphere J and any $\epsilon > 0$, there is an unknotted polygon P (the boundary of a polyhedral disk) and a homeomorphic map of J onto P under which every two corresponding points are at distance less than ϵ apart.

S. S. Cairns (Urbana, Ill.).

White, Paul A. Regular convergence in terms of Čech cycles. Ann. of Math. (2) 55, 420-432 (1952).

As indicated in the title, the author formulates the concept of regular convergence in terms of Čech cycles and succeeds to a high degree in establishing the major theorems concerning regular convergence of sequences of sets using the covering techniques familiar in the Čech homology theory. Thus these results are freed from dependence on a metric, which was inherent in their original form due to the use of Vietoris cycles. In addition, however, the author proves some new results, the most noteworthy of which is to the effect that if a sequence $[A_i]$ converges regularly to a limit set, then corresponding to any open covering of the space there exists a refinement that is normal with respect to Čech cycles on A_i simultaneously for all i .

G. T. Whyburn.

Wallace, A. D. The map excision theorem. Duke Math. J. 19, 177-182 (1952).

Soient X et Y des espaces pleinement normaux (fully normal), c'est-à-dire paracompacts au sens de Dieudonné. Soient $A \subset X$, $B \subset Y$, A et B fermés. Soient $H^p(X, A)$, $H^p(Y, B)$ les groupes de cohomologie relatifs d'Alexander-Kolmogoroff-Spanier. Soit f une application continue de X dans Y , telle que $f(A) \subset B$; supposons que f transforme les fermés en fermés, et que la restriction de f à $X - A$ soit un homéomorphisme sur $Y - B$. L'auteur démontre que f définit un isomorphisme $H^p(Y, B) \approx H^p(X, A)$.

H. Cartan (Paris).

Reichelderfer, Paul V. On the barycentric homomorphism in a singular complex. Pacific J. Math. 2, 73-97 (1952).

This paper extends T. Rado's investigation of 'inessential identifications' in singular homology theory. [Revista Mat.

Univ. Parma 2, 3-18 (1951); Pacific J. Math. 1, 265-290 (1951); these Rev. 13, 373]. The integral chain groups C_p of Rado's version of the theory are free groups, generated by cells (v_0, \dots, v_p, T) where (v_0, \dots, v_p) is an ordered set of (not necessarily distinct) vertices in separable Hilbert space, and T is a continuous map of the convex hull of (v_0, \dots, v_p) into a fixed topological space X . Let N_p be the kernel of the barycentric homomorphism of C_p into itself, where this homomorphism is defined in the natural way. The principal result: The natural homomorphisms of the homology groups of $\{C_p\}_p$ into those of $\{C_p/N_p\}_p$ are isomorphisms onto. A sketch is given of the proof of another theorem of the same type. Let A_p be the subgroup of C_p generated by all chains of the form $(v_0, \dots, v_p, T) - (w_0, \dots, w_p, TS)$, where w_0, \dots, w_p are linearly independent points of the Hilbert space and S is the affine map such that $S(w_i) = v_i$ for $i = 1, \dots, p$. Let M_p be the division hull of $A_p + N_p$. Then the homology groups of $\{C_p/M_p\}_p$ are isomorphic to those of $\{C_p\}_p$. The paper also contains a detailed investigation of the algebraic relations between the barycentric, transposition, cone, and barycentric homotopy operators. *J. L. Kelley.*

Zimmermann, Wolfhart. Eine Cohomologietheorie topologischer Räume. Math. Z. 55, 125-166 (1952).

The author generalizes the Čech and Alexander definitions of cohomology as follows. The concept of complex is extended to mean an arbitrary collection of (not necessarily finite) subsets of some set. Cohomology groups are defined for such complexes, and it is shown how mappings between these complexes induce homomorphisms of the cohomology groups. To any covering of a topological space R there are associated two complexes, the first is the nerve of the covering, and the second is that collection of subsets of R each of which is contained in some element of the covering. Given a family of coverings directed by refinement, a limiting process applied to the first type of complex gives rise to cohomology groups of R which generalize the Čech groups and applied to the second type of complex leads to groups which generalize the Alexander groups. A cofinal subfamily of coverings gives rise to isomorphic groups so that for compact spaces the groups based on the family of all open coverings are isomorphic to the classical Čech and Alexander groups. No examples are computed nor are any applications of the new groups given. *E. H. Spanier.*

Čogošvili, G. S. On the equivalence of the functional and spectral theory of homology. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 421-438 (1951). (Russian)

The author's aim is to prove isomorphism between the spectral homology groups of a (locally compact Hausdorff) space (i.e. groups based on the nerves of coverings or partitions (see below)) and the functional groups (based on set functions) introduced by Kolmogorov and Alexander. In

the course of the argument a large number of types of homology groups, whose definitions differ slightly from each other, are considered. In particular, there are the groups, introduced by the author, based on partitions: A partition is a finite collection of mutually disjoint sets, whose union is the whole space; the nerve of the partition is a complex, whose vertices are these sets, where a collection of vertices forms a simplex, if the closures of the corresponding sets have a non-empty intersection. As usual, refinements determine simplicial maps of the nerves; the advantage of partitions against coverings is that these maps are uniquely determined (and not only their homology type). The order of singularity s of a set in a partition (or covering) is the length of the shortest chain (in the sense of the closures of any two adjacent sets intersecting) of sets of the partition, whose last term has noncompact closure. Considering the nerve modulo the subcomplex, consisting of the simplices, all of whose vertices have order of singularity $\leq s$, one gets different groups. Groups of the space are now obtained by taking inverse limits (compact coefficient group), either of the homology groups of the nerves, or of the cycles, defining a limit group of boundaries properly, etc. All these groups are shown isomorphic, for all values of $s \geq 1$. They are also isomorphic with the groups based on closed coverings; the proof starts from the fact that the closures of the sets in a partition form a closed covering.

In the Alexander-Kolmogorov scheme an r -chain is a function of $r+1$ variable sets (with compact closures), with values in a compact group, satisfying three conditions: (1) skew symmetry, (2) additivity in each variable, (3) the function value is 0 if the closures of the argument sets have empty intersection; a boundary operator is suitably defined. (A similar type of chains, using arbitrary sets as arguments, are introduced by the author, and proved equivalent to the Čech groups based on finite coverings. The author mentions that some criticism [these Rev. 3, 142; Fiz.-Mat. Ref. Zhurnal 5, 118 (1941)] of an earlier paper of his on the same subject, is unjustified since already there he was using partitions instead of coverings). It is now proved that the Alexander-Kolmogorov groups are isomorphic with the groups based on closed coverings and order of singularity 1 ("Alexandrov-groups") and therefore all the other groups of §1. This is proved by setting up a chain isomorphism between the Alexander-Kolmogorov-chains, and one of the chain groups of §1, of order of singularity 3. Each Alexander-Kolmogorov-chain determines a chain on the nerve of each partition, and so one has a natural map. The proof that this is an isomorphism involves a somewhat complicated combinatorial argument; in particular, a certain scheme is adopted to associate a partition with an arbitrary collection of sets; roughly speaking, any set in the partition consists of those points which belong exactly to p definite sets of the collection ($p = 1, 2, \dots$). *H. Samelson.*

GEOMETRY

✓Hilbert, D., and Cohn-Vossen, S. Geometry and the imagination. Translated by P. Neményi. Chelsea Publishing Company, New York, N. Y., 1952. ix+357 pp. \$5.00.

Translated from Anschauliche Geometrie, Springer, Berlin, 1932.

★Neiss, Fritz. Analytische Geometrie. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1950. viii+167 pp. 9.60 DM.

A textbook intended as a companion to the author's Determinanten und Matrizen [Springer, Berlin, 1948; these Rev. 11, 153]. The chapter headings are as follows: 1)

Elementare Einführung; 2) Geometrie der Geraden und Ebene unter Benutzung der Vektorrechnung; 3) Kongruente und ähnliche Abbildungen; 4) Projektive Geometrie der linearen Gebilde; 5) Kurven zweiter Ordnung; 6) Flächen zweiter Ordnung.

Breidenbach, W. Über ähnliche seitengebundene Dreiecke. Math.-Phys. Semesterber. 2, 231-237 (1952).

Godeaux, Lucien. Sur les tétraèdres de Moebius. Bull. Soc. Roy. Sci. Liège 20, 464-470 (1951).

Ölander, V. R. Quelques méthodes et formules pour calculer l'excès sphérique des polygones plus étendus. Veröff. Finn. Geodät. Inst. 36, 163-176 (1949).

The author obtains three different formulae for the area of a geodesic polygon on a spheroid, in terms of the latitudes ϕ_i and longitudes λ_i of the vertices. He applies these to a dodecagon formed by twelve actual places in Finland, obtaining results which differ from $46''.46$ by less than $0''.01$, in angular measure. He compares this value with the spherical excess $45''.75$, obtained by direct measurement of the angles, and attributes the difference to the eccentricity of the earth's shape. *H. S. M. Coxeter.*

Witt, Ernst. Ein kombinatorischer Satz der Elementargeometrie. Math. Nachr. 6, 261-262 (1952).

The author proves a generalisation of van der Waerden's theorem. The author was unaware of a paper by R. Rado [Proc. London Math. Soc. (2) 48, 122-160 (1943); these Rev. 5, 87] in which it is stated that the result is due to T. Grünwald (Gallai) [loc. cit. p. 128 and 130]. The various methods of proof are closely related. *P. Erdős.*

Obláth, Richárd. Une remarque sur la théorie des constructions géométriques. Mat. Lapok 2, 219-221 (1951). (Hungarian. Russian and French summaries)

The author proves that every construction which can be done by ruler and compass can be carried out by the ruler alone, if an arc of a circle (without its center) and the trisecting points of the arc are given. *P. Erdős.*

Bachmann, Friedrich. Über die Konstruierbarkeit mit Lineal, Rechtwinkelmass und Eichmass. Math.-Phys. Semesterber. 1, 77-88 (1949).

According to Hilbert [Grundlagen der Geometrie, 7th ed., Teubner, Leipzig, 1930, ch. VII] all constructions by ruler and compass which can be performed on the basis of his axioms I1-I3, II, III and the Parallel axiom IV can be carried out by using the ruler and a unit length. The ruler is used 1) to connect two points, 2) to find the intersection of two non-parallel lines. The unit length may be laid off a given line from a given point toward a given side.

The present paper admits the ruler only for 1) and replaces the parallel axiom by the (according to a result of Dehn) weaker axiom that the fourth angle in a quadrangle with three right angles is a right angle. On the other hand, the "angle iron" is used permitting construction of the perpendicular through a given point to a given line. It is shown that the four basic elementary constructions of Hilbert can be carried out by these instruments. However, not every construction possible under Hilbert's hypothesis is possible here. The algebraic reason for this is that all "numbers" constructible from the unit length are obtained by repeated use of the operations $u \pm v$, $(u^2 + v^2)^{1/2}$,

$uv(u^2 + v^2)^{-1/2}$, but there is no general construction for product and quotient. *H. Busemann (Auckland).*

Bachmann, Friedrich. Zur Begründung der Geometrie aus dem Spiegelungsbegriff. Math. Ann. 123, 341-344 (1951).

A. Schmidt gave a system of axioms for absolute geometry based on line reflection only [Math. Ann. 118, 609-625 (1943); these Rev. 6, 13]. The present paper reduces this system as follows. A group which can be generated by involutions a, b, \dots corresponding to line reflections is given. The reflection a is called the line a . Define $a \perp b$ (a perpendicular to b) if $a \neq b$ and $ab = ba$. If $a \perp b$ then $P = ab$ is a point reflection and called the point P . The point P is incident with the line g if $P \neq g$ and $Pg = gP$. The further axioms are: Two points are incident with exactly one line. Through a given point there is a perpendicular to a given line. The product of the reflections in three concurrent lines is a line reflection. The product of the reflections in three lines perpendicular to the same line is a line reflection. There are three lines g, h, j such that $g \perp h$, j does not pass through $g \perp h$, and j is not perpendicular to either g or h . *H. Busemann (Auckland).*

Zacharias, Max. Die ebenen Konfigurationen (10₃). Math. Nachr. 6, 129-144 (1951).

The author has rediscovered seven of the ten configurations 10₃ described by S. Kantor [S.-B. Math.-Nat. Cl. Akad. Wiss. Wien 84, Abt. 2, 1291-1314 (1882)].

H. S. M. Coxeter (Toronto, Ont.).

Skornyakov, L. A. The configuration D_9 . Mat. Sbornik N.S. 30(72), 73-78 (1952). (Russian)

Theorem D_9 is the special case of the Theorem of Desargues in which two vertices of one of the given triangles lie on sides of the other. The little Theorem of Desargues (Theorem D_{10}) is the special case of the Theorem of Desargues in which the center of perspectivity lies on the axis of perspectivity. It was shown by Moufang [Math. Ann. 106, 755-795 (1932)] that D_9 implies D_{10} if the diagonal points of a quadrilateral are not collinear. D_{10} is equivalent to coordinatization from an alternative field. In an effort to free this result from the assumption on quadrilaterals, the author derives a number of theorems which are always consequences of D_9 and shows that several of these are equivalent to D_9 . *Marshall Hall (Washington, D. C.).*

Skornyakov, L. A. Projective planes. Uspehi Matem. Nauk (N.S.) 6, no. 6 (46), 112-154 (1951). (Russian)

This paper presents an excellent survey of the literature on projective planes and summarizes the present state of knowledge. The sections are: 1) Fundamental concepts, 2) Free planes, 3) The ternary ring, 4) Configurations, 5) Collineations and correlations, and a final Supplement. A number of the most basic theorems are proved. It is shown that subplanes of a free plane are free and that every plane is a homomorphic image of a free plane. By a homomorphism is meant a many-to-one mapping preserving incidences, but not, of course, unions and intersections. The relation of the natural ternary ring to local and projective configurations is well treated, including the equivalence of the little Theorem of Desargues to the existence of coordinates from an alternative division ring. The equivalence of the Theorem of Desargues to the existence of certain collineations is given. Finally the results on cyclic planes, the nonexistence of finite planes with $n^2 + n + 1$ points for certain

values of n , and topological planes are mentioned briefly. There is an extensive bibliography and an assortment of problems and examples is included. *Marshall Hall.*

Baker, H. F. Note on the foundations of projective geometry. *Proc. Cambridge Philos. Soc.* 48, 363-364 (1952).

Es wird bewiesen, dass der Schnittpunktsatz von Pappus-Pascal äquivalent ist mit dem Schnittpunktsatz: Wenn die Seiten EF , FD , DE eines Dreiecks DEF die Gerade durch zwei beliebige Punkte I , J resp. in den Punkten P , Q , R schneiden, und wenn $(IJPP')$, $(IJQQ')$, $(IJRR')$ harmonische Quadrupel sind, so gehen die Geraden DP' , EQ' , FR' durch einen Punkt. *R. Moufang (Frankfurt a.M.).*

Moser, Leo. On the different distances determined by n points. *Amer. Math. Monthly* 59, 85-91 (1952).

Denote by $f(n)$ the least number of different distances determined by n points in the plane. The reviewer [same *Monthly* 53, 248-250 (1946); these *Rev.* 7, 471] has shown that

$$(n-1)^{1/2} - 1 < f(n) < en/[\log n]^{1/2}.$$

It has been conjectured that $f(n) > n^{1-\epsilon}$ for every ϵ and n sufficiently large. The author proves that

$$f(n) > n^{2/3}/2 \cdot 9^{1/2} - 1.$$

If the n points form a convex polygon then the author proves that the least number $f^*(n)$ of different distances is $\geq [\frac{1}{2}(n+2)]$. It has been conjectured that $f^*(n) = [\frac{1}{2}n]$.

P. Erdős (Los Angeles, Calif.).

Convex Domains, Extremal Problems

Sengenhorst, Paul. Über konvexe Funktionen. *Math.-Phys. Semesterber.* 2, 217-230 (1952).
Expository paper.

Eggleston, H. G. Measure of asymmetry of convex curves of constant width and restricted radii of curvature. *Quart. J. Math., Oxford Ser. (2)* 3, 63-72 (1952).

Adding to results of Besicovitch [*J. London Math. Soc.* 23, 237-240 (1948); 26, 81-93 (1951); these *Rev.* 10, 320; 12, 850] the author proves the following conjecture of Besicovitch. Let Δ be a convex plane set and let

$$g(\Delta) = 1 - (\text{area of largest symmetric subset of } \Delta) / (\text{area of } \Delta).$$

Suppose that for a number d , $\frac{1}{2} \leq d \leq 1$, Ω_d is the set of points at distance $\leq 1-d$ from the Reuleaux triangle of constant width $2d-1$. Then for every set Δ of constant width 1 such that the curvature at every point of the boundary of Δ is between d and $1-d$ the measure of asymmetry $g(\Delta) \leq g(\Omega_d)$.

M. M. Day (Urbana, Ill.).

Santaló, L. A. Generalization of an inequality of H. Hornich to spaces of constant curvature. *Revista Unión Mat. Argentina* 15, 62-66 (1951). (Spanish)

Let C be a rectifiable curve of length L in an n -dimensional space of curvature k^2 . The volume V of the set of points with distance $\leq R$ from C satisfies the inequality

$$(*) \quad V \leq k^{-n+1} \left(\lambda_{n-1} L \sin^{n-1} kR + n \lambda_n \int_0^R \sin^{n-1} kt \, dt \right)$$

where $\lambda_i = \pi^{i/2} / \Gamma(\frac{1}{2}i+1)$. (If $k^2 < 0$ then $\sin it = i \sin ht$, $\cos it = \cos ht$.) The equality sign holds in (*) if and only if

the centers of the spheres of radius R which intersect C more than twice form a set of measure 0, and if in addition C , when open, lies in every sphere of radius R that contains the endpoints of C . *H. Busemann (Auckland).*

Besicovitch, A. S. Variants of a classical isoperimetric problem. *Quart. J. Math., Oxford Ser. (2)* 3, 42-49 (1952).

The author gives a new proof of the known inequality $L^2 - 4\pi T \geq (2\pi R - L)^2$, where L , T and R denote the perimeter, area and radius of the circumcircle of a convex domain. Then he gives the complete solutions of the two following problems: Find the convex domain of given perimeter contained in a convex region, or containing a convex one, respectively, and having the largest possible area.

L. Fejes Tóth (Veszprém).

Hammersley, J. M. The total length of the edges of the polyhedron. *Compositio Math.* 9, 239-240 (1951).

Let L denote the sum of the lengths of the edges of a convex polyhedron containing a sphere of unit diameter. It has been conjectured by the reviewer [*Norske. Vid. Selsk. Forh., Trondhjem* 21, 32-34 (1948); these *Rev.* 11, 386] that $L \geq 12$ with equality only for the cube, and that for triangularly faced polyhedra $L \geq 216^{1/3}$ with equality only for the regular tetrahedron or octahedron. The author proves that for polyhedra having at most n -sided faces $L > 10(\pi n \tan \pi/n)^{1/2} / 3 > 10\pi/3 = 10.47 \dots$. For the case when no restriction is made on the number of the sides, this inequality is a little sharper than the inequality given by the reviewer, $L > 10.46$. *L. Fejes Tóth (Veszprém).*

Hadwiger, H. Translative Zerlegungsgleichheit k -dimensionaler Parallelotope. *Collectanea Math.* 3, 11-23 (1950).

As proved by A. Erch [Comment. Math. Helv. 18, 224-231 (1946); these *Rev.* 8, 83] any two k -dimensional parallelotopes of equal volume may be decomposed into finitely many mutually congruent polyhedra. The author proves that in this statement the word 'congruent' may be replaced by 'congruent by translation', and adds remarks on the introduction of k -dimensional volume. *B. Jessen.*

Rogers, C. A. The closest packing of convex two-dimensional domains. *Acta Math.* 86, 309-321 (1951).

The main result of the present paper is the following remarkable theorem. If n similarly situated congruent convex discs of area a can be packed into a convex domain of area A , then $na/A \leq nd/(n-1+d)$, where d denotes the density of the closest lattice packing of the discs. It follows that the density of an irregular packing of an infinite set of homothetic congruent convex discs cannot exceed the density of the closest lattice of the discs. If the discs have a center of symmetry, then by a result of the reviewer [*Acta Sci. Math. Szeged* 12, Pars A, 62-67 (1950); these *Rev.* 12, 352] the last proposition holds without restriction of homothetic discs. *L. Fejes Tóth (Veszprém).*

Massera, J. L., and Schäffer, J. J. Minimum figures covering points of a lattice. *Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística* 2, 55-74 (1951). (Spanish. English summary)

Let L be the lattice of all parts with integral coordinates in the cartesian plane. A set M is called covering if any set obtained from M by a motion of the plane contains at least one point of L . Santaló put the problem to find the greatest

lower bound of the measures of all covering sets. Here various covering sets are constructed, in particular a convex set C of area $4/3$. It is shown that any convex covering set has at least area $2 - 2^{-1/2}$ and it is conjectured that C yields the minimum for convex sets. A sufficient condition for a convex set to be covering is derived, which is also necessary when the diameter of the set is less than $5^{1/2}$. A necessary and sufficient condition is given for a convex covering set not to contain a proper convex subset which is still covering.

H. Busemann (Auckland).

Bang, Thøger. A solution of the "plank problem." Proc. Amer. Math. Soc. 2, 990-993 (1951).

A simplified and generalized version of the author's previous proof [Mat. Tidsskr. B. 1950, 49-53; these Rev. 12, 352] of Tarski's "plank problem". The review of the previous paper applies, even more appropriately, to the present paper.

W. Gustin (Bloomington, Ind.).

Algebraic Geometry

Tosi, Armida. Sulle curve del 4° ordine intersezioni di quadriche di rotazione. Period. Mat. (4) 30, 33-41 (1952).

Chisini, O. Singolarità delle curve algebriche plane. Period. Mat. (4) 29, 142-166 (1951).

Eine sehr übersichtliche Zusammenstellung der klassischen Sätze über die Singularitäten ebener Kurven für solche Leser, die in das Gebiet erst eindringen wollen. Speziellere Abschnitte, die beim ersten Lesen überschlagen werden können, sind, wie in einführenden Lehrbüchern, mit einem Stern bezeichnet. So gibt Verf. zunächst die Definition des singulären Punktes mit Hilfe der partiellen Ableitungen, insbesondere des Doppelpunktes und der Spitze; Singularitäten der allgemeinen Kurve eines Büschels und Satz von Bertini; Sätze über Geschlecht und Polaren; Riemannsche Flächen und (lineare und superlineare) Zweige; Auflösung der Singularitäten durch quadratische Cremona-Transformationen. In 9 geschickt gewählten Beispielen, 8 Quartiken und der Asteroide, bestimmt er Lage und Art der Singularitäten. Sein Bestreben ist dabei, durch geschicktes Einbetten in Büschel das formale Rechnen möglichst zu vermeiden. Die Sätze der Abschnitte sind so, wie sie dastehen, nicht richtig; doch ist die Berichtigung leicht zu finden.

O. H. Keller (Halle).

Zappa, Guido. Sul limite di una serie lineare d'una curva irriducibile tendente ad una riducibile. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 13(82), 35-54 (1949).

A plane algebraic curve C , consisting of two components C_1 and C_2 , is the limit of a plane algebraic curve \bar{C} of genus p . (It is apparently assumed that \bar{C} , C_1 and C_2 have no singularities other than ordinary double points, and that C_1 and C_2 have no contact.) The common points of C_1 and C_2 consist partly of limits of double points of \bar{C} , and partly of a group H of "points of connection" of C : we regard as virtually non-existent the double points of C_1 and C_2 which are not limits of double points of \bar{C} . It is shown that a linear series $g(n, r)$, of order n and dimension r , of \bar{C} has as limit a series $g(n, r)$ of C , which is the sum of a finite number of series of equivalence, $g_i(n, r)$, on C , where

$g_i(n, r)$ has as its components on C_1 and C_2 linear series $g_i(s_i, u_i)$, $g_i(t_i, v_i)$ ($s_i + t_i = n$). A number of relations are given between the various u 's and v 's and also between the consecutive s 's and t 's, and particular attention is paid to the case where H consists of only a single point (though even in this case the detailed results are too lengthy to report). The paper concludes with a lengthy result on the limit of the canonical series in the general case, preceded by some simpler results for the case of a single point of connection which is not a Weierstrass point for either C_1 or C_2 . It is suggested that the case of more than two distinct components will give nothing essentially new, but that multiple components would give rise to serious difficulties.

D. B. Scott (London).

Burniat, Pol. Sur les surfaces canoniques de genres $p_0 = p_\infty = 4$, $p^{(1)} \geq 11$. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 367-377 (1951).

Ce mémoire fait suite à deux autres sur le même sujet [Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 880-896 (1950); (5) 37, 241-251 (1951); ces Rev. 13, 63]. L'auteur reprend, au début, la méthode par laquelle il ramène la construction des surfaces doubles canoniques à celle des surfaces rationnelles F^n normales dans S_3 à sections C d'ordre n et de genre $\bar{\omega} = n - 2$, portant un système effectif, non composé avec un faisceau, de courbes $4C - 2C'$ (où $|C'|$ est l'adjoint de $|C|$). Ces surfaces canoniques ont le genre linéaire $p^{(1)} = 2n + 1$, et le présent mémoire est consacré au cas des surfaces F^{10} canoniques de genre $p^{(1)} = 11$, obtenues à partir des doubles des surfaces F^3 à sections de genre 3. 1) Lorsque F^3 admet une droite triple D , la surface double canonique dépend de 22 modules, et on obtient ainsi des surfaces canoniques F^{10} ayant D sextuple et cinq coniques doubles bisécantes à D . Leur classe dépend de 28 modules. 2) Lorsque F^3 admet trois droites doubles D_1, D_2, D_3 formant un trièdre dont le sommet O est triple pour F^3 , on obtient des surfaces doubles canoniques dépendant de 24 modules. Elles sont limites de surfaces canoniques F^{10} admettant D_1, D_2, D_3 quadruples, une droite double dans chaque face de ce trièdre et une biquadratique double bisécante à D_1, D_2, D_3 . La classe d'une telle surface dépend de 27 modules. 3) Lorsque F^3 admet une cubique gauche double C , on obtient des surfaces doubles canoniques dépendant de 23 modules, qui ne sont limites d'aucune surface canonique simple.

Les surfaces simples F^{10} des deux premiers types dépendent de moins de 30 modules, ce qui confirme une critique émise sur cette question par Zariski [Algebraic Surfaces, Springer, Berlin, 1935].

L. Gauthier (Nancy).

Roth, L. Some threefolds on which adjunction terminates. Proc. Cambridge Philos. Soc. 48, 233-242 (1952).

Après avoir remarqué que les variétés algébriques à trois dimensions connues sur lesquelles la construction de l'adjonction se termine possèdent un système anticanonique, l'auteur s'occupe des variétés algébriques V à trois dimensions qui possèdent un tel système anticanonique effectif $|A|$; cela signifie que le système canonique $|K|$ de V , construit comme système $|F' - F|$ à partir d'un système linéaire $|F|$ de surfaces donné sur V et de son adjoint $|F'|$, est virtuel. Le système anticanonique est donné par $|A| = |F - F'|$. Pour plus de simplicité l'on peut supposer que la variété V soit dépourvue de points singuliers. Alors, de l'existence du système $|A|$ on déduit: que le genre géométrique et les plurigenres de V sont tous nuls; que sur V la succession des systèmes adjoints successifs d'un quel-

conque système $|F|$ de surfaces est terminée; que dans le cas où la surface générale du système $|A|$ est irréductible elle a tous ses genres égaux à un et qu'elle ne possède pas de courbes exceptionnelles; que si $|A|$ a une dimension ≥ 3 et sa surface générale est régulière, alors l'irrégularité superficielle de V est nulle. Après ces considérations générales, l'auteur se place dans le cas où V est complètement régulière, et son système anticanonique $|A|$ est irréductible et dépourvu de points-base. Il y a alors trois cas à considérer: le cas où la série caractéristique de $|A|$ est simple et le système caractéristique de $|A|$ est irréductible; le cas où la série caractéristique de $|A|$ est composée; et enfin le cas où le système caractéristique de $|A|$ est composé. Dans le premier cas la dimension de $|A|$ est $p+1$, où p signifie le genre virtuel du système caractéristique de $|A|$; on a dans ce cas $p \geq 3$; et V se représente birationnellement sur une W^{2p-3} de S_{p-1} dont les courbes-section sont des courbes canoniques de genre p ; ce sont des variétés W^{2p-3} étudiées par Fano (elles existent seulement pour $p \leq 37$). Dans le deuxième cas les courbes caractéristiques de $|A|$ sont hyperelliptiques; et la série caractéristique de $|A|$ est composée avec une involution I_2 . On a encore pour la dimension de $|A|$ la valeur $p+1$. La variété V se représente dans ce cas sur une W^{p-1} double de S_{p-1} dont les courbes-section sont rationnelles. Dans le troisième cas la courbe caractéristique générale de $|A|$ se compose d'un certain nombre n de courbes elliptiques appartenant à une congruence linéaire rationnelle Γ ; le nombre n peut avoir seulement les valeurs 1, 2, 3, 4. Il n'est pas possible référer ici tous les détails des discussions, et les exemples que l'auteur donne à la fin. Comme conclusion de cette recherche, et d'une autre qui va être bientôt publiée, on trouve que les variétés à trois dimensions complètement régulières sur lesquelles la construction de l'adjonction se termine peuvent être soit birationnelles, soit unirationnelles, soit irrationnelles. *E. G. Togliatti.*

d'Orgeval, B. Le diviseur de Severi des surfaces régulières. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 495-512 (1950).

Etendant un résultat de Commessatti [Rend. Sem. Mat. Univ. Padova 1, 1-45 (1930)], l'auteur obtient une limitation du diviseur d'une surface algébrique dans le cas où le genre linéaire de la surface supposée régulière est inférieur au triple du genre arithmétique augmenté de 4. Il en résulte que dans certains cas on peut assurer la cyclicité du groupe de la division, ceci en supposant que le système canonique satisfait à certaines hypothèses simples. (Author's résumé.)

D. B. Scott (London).

Berzolari, Luigi. Sulle normali delle varietà algebriche. Rend. Accad. Naz. dei XL (4) 1, 23-29 (1950).

Let V_k^n be an arbitrary irreducible variety of dimension k and order n defined over the complex field, and let S_r be its ambient projective space. Let a metric be defined in S_r by means of an improper hyperplane χ and an $(r-2)$ -dimensional quadric Θ in χ , and assume that χ does not contain any space S_k tangent to V_k^n . With orthogonality defined in the usual way by means of the polarity Θ , let $\varphi(k, n, r)$ denote the number of spaces S_{r-k} on a generic point P of S_r , which are orthogonal to V_k^n . It is shown that $\varphi(k, n, r) = n + \rho_1 + \rho_2 + \dots + \rho_k$, where ρ_i is the i th class of V_k^n in the sense of Severi [Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (2) 52, 61-118 (1903)]. This is done by relating $\varphi(k, n, r)$ to the function $\varphi(k-1, n, r-1)$ for a general hyperplane section of V_k^n , and evaluating $\varphi(1, n, r-k+1)$ with the aid of the Cayley-Brill correspondence formula.

The case of a general hypersurface V_{r-1}^n and that of a complete intersection V_{r-2}^n of two hypersurfaces are examined in greater detail. [We call attention to a typographical error in equation (3) page 26. It should read $\varphi(k, n, r) = \varphi(1, n, r-k+1) + \rho_1 + \rho_2 + \dots + \rho_k$.]

H. T. Muhly (Iowa City, Iowa).

Villa, Mario. Caratterizzazioni differenziali di enti algebrici. Rend. Sem. Mat. Fis. Milano 21 (1950), 51-58 (1951).

A summary of results on the problem of characterizing the varieties of Veronese and Segre in terms of quasi-asymptotics is given. An extensive bibliography is included.

H. T. Muhly (Iowa City, Iowa).

Godeaux, Lucien. Sur certaines transformations monoidales et leur représentation. Bull. Soc. Roy. Sci. Liège 20, 143-157 (1951).

Le présent mémoire concerne l'application aux transformations monoidales d'un mode de représentation des transformations birationnelles du plan et de l'espace, par une surface ou une V_3 , exposé dans divers travaux antérieurs [cf. spécialement Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° (2) 24, no. 2 (1949); ces Rev. 11, 739; qui contient un exposé d'ensemble et toute la bibliographie]. Dans une première partie l'auteur étudie complètement les éléments fondamentaux d'une transformation monoidale T entre deux espaces projectifs Σ, Σ' , tant dans Σ que dans Σ' . Ceci lui permet d'établir diverses relations d'équivalences sur la variété V représentative de T . Si les gerbes homologues dans T se correspondent dans une transformation birationnelle d'ordre n , et si le système homaloïdal de Σ est d'ordre $n+m$, on obtient une variété image V appartenant à un S_r et l'étude faite permet de tirer l'inégalité remarquable: $r \leq m+n+9$.

La fin du mémoire est consacrée à l'étude du cas où les gerbes homologues dans T se correspondent homographiquement. Si O, O' sont les sommets de ces gerbes, on obtient alors le résultat suivant: La variété V est d'ordre $6n+2$ et elle contient un réseau de surfaces F_0 rationnelles d'ordre $2n+2$ se coupant deux à deux suivant des coniques. Les surfaces F qui sur V correspondent aux plans de Σ , et les surfaces F' qui correspondent aux plans de Σ' , s'obtiennent en ajoutant à $|F_0|$ soit la surface G qui correspond au point fondamental O de Σ , soit la surface G' qui correspond au point fondamental O' de Σ' .

L. Gauthier (Nancy).

Fernández Biarge, Julio. Remark on the coincidences of a birational correspondence. Revista Mat. Hisp.-Amer. (4) 11, 288-290 (1951). (Spanish)

The author refers to a previous paper of his [Revista Mat. Hisp.-Amer. (4) 10, 3-11 (1950)] and to some criticism which appeared against it in Zentralblatt für Mathematik 38, 317 (1951), in these Rev. 12, 529 (1951), and in a paper by F. Gaeta [Revista Mat. Hisp.-Amer. (4) 11, 132-137 (1951); these Rev. 13, 679]. He recognizes that, to meet such criticism, it suffices to suppress the lines from the 4th to the 8th of the last page of his paper.

B. Segre.

Dedd, Modesto. Determinazione topologica di molteplicità. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 79-90 (1950).

Let us consider a self-corresponding point in a Cremona transformation between two superposed planes. The multiplicity of such a point is usually defined using isologic curves. The author gives a topological definition of the multiplicity,

first as a topological invariant of a point with respect to a three-fold manifold (closed, oriented) in a space S_4 , second as a topological invariant of two (closed, oriented) curves in a space of three dimensions. From this a criterion for the evaluation of the multiplicity is derived analogous to that of Halphen-Zeuthen for the multiplicity of a point of intersection of two plane curves. The coincidence of these definitions with the usual one is shown. The results hold also for transformations more general than the Cremona transformations.
E. Bompiani (Rome).

Hodge, W. V. D. Tangent sphere-bundles and canonical models of algebraic varieties. J. London Math. Soc. 27, 152-159 (1952).

Let $H(m, s)$ be the Grassmann manifold of all subspaces of dimension m in a complex vector space of dimension $m+s$. The reviewer proved that the tangent sphere bundle of a compact complex manifold M of dimension m is induced by a mapping $f: M \rightarrow H(m, s)$, if $m \leq s$, and that f is defined up to a homotopy [Ann. of Math. (2) 47, 85-121 (1946); these Rev. 7, 470]. For applications to algebraic geometry it is advantageous to replace $H(m, s)$ by the Grassmannian $G_{m,s}$ of all $(m-1)$ -dimensional linear spaces in a complex projective space of dimension $m+s-1$, and f by a mapping $f^*: M \rightarrow G_{m,s}$. The purpose of this paper is to determine the homotopy class of mappings f^* in case M is a non-singular algebraic variety. Since the Schubert varieties form a homology base on $G_{m,s}$, $f^*(M)$ is homologous to a linear combination of them. It is proved that the coefficients of this linear combination can be expressed in terms of the intersection numbers of the canonical systems of M . The mapping f^* can sometimes be taken to be a rational transformation, for instance, in the case that on M there are g linearly independent Picard integrals of the first kind $du^i = A_{\alpha}^i ds^{\alpha}$, $i=1, \dots, g$, $\alpha=1, \dots, m$, such that the matrix (A_{α}^i) is of rank m everywhere. The construction of f^* in the particular case of curves is carried out in detail.

S. Chern (Chicago, Ill.).

Cherubino, Salvatore. Sui periodi degli integrali multipli delle varietà algebriche. Rivista Mat. Univ. Parma 2, 175-194 (1951).

The formalism of the theory of Riemann matrices is applied to the matrix of periods of the differentials of first kind of any given degree on an algebraic variety. The results obtained all follow directly from the Hodge theory. The second part of the paper considers two such period matrices (got from different varieties) that are connected by the "relations of Hurwitz". Some simple properties of these "correspondences" are deduced, but no geometric interpretation is given.
M. Rosenlicht (Princeton, N. J.).

Differential Geometry

van der Waag, E. J. Équivalence d'arc et de corde. I. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 390-403 (1951).

φ, ψ : fonctions continues définies sur un intervalle $[a, b]$; $\bar{\varphi}(t)$: dérivé supérieur de φ en t ; $\varphi(t)$: dérivé inférieur de φ en t ; $\varphi^*(t)$: nombre quelconque entre $\bar{\varphi}(t)$ et $\varphi(t)$. Le §1 contient des généralisations du théorème de la moyenne, ainsi: (1, 3) Sous les hypothèses $\psi(b) \neq \psi(a)$, $\psi^*(t) \neq 0$, $-\infty < \varphi^*(t) < +\infty$, $-\infty < \psi^*(t) < +\infty$ pour $a < t < b$ quels

que soient $\varphi^*(t)$ et $\psi^*(t)$, il existe un ξ entre a et b tel que par un choix convenable de $\varphi^*(\xi)$ et de $\psi^*(\xi)$, le rapport $[\varphi(b) - \varphi(a)]/[\psi(b) - \psi(a)]$ soit $= \varphi^*(\xi)/\psi^*(\xi)$. Les propositions classiques sur la dérivation d'une somme, d'un produit et d'un quotient sont généralisées de manière similaire. Ces résultats préliminaires sont utilisés dans le §2 pour établir sobrement et élégamment les théorèmes sur les nombres dérivés généralisés énoncés sans démonstration par le rapporteur dans sa Thèse [Actualités Sci. Ind., nos. 885 et 886, Hermann, Paris, 1941; ces Rev. 7, 67] pp. 98-99, 105-106, 119-120. L'exemple $\varphi(t) = t^2(t)$ où ξ est la fonction introduite par Lebesgue dans ses "Leçons sur l'Intégration" [Gauthier-Villars, Paris, 1928] p. 56 infirme deux assertions analogues de cette Thèse, à savoir p. 99, 2) et p. 106, 4b). [Remarques du rapporteur. (i) Les équivalences incriminées deviennent exactes si φ est supposée absolument continue. (ii) Les §§1 et 2 ont des points en commun avec la Dissertation de F. Alt [Vienne, 1932] et un article de Jarnik [Časopis Pěst. Mat. Fys. 74, D37-D51 (1949); ces Rev. 11, 540].] (§3) En un point O d'une courbe rectifiable k l'arc est équivalent à la corde au sens ordinaire lorsque le rapport $(\text{arc } AB)/(\text{corde } AB)$ tend vers 1 quand $B=O$ et A tend vers O . L'équivalence est dite "au sens contrastant" si A et B tendent vers O de part et d'autre, "uniforme" si A et B tendent librement vers O . Si k admet en O une tangente ordinaire, les deux premières équivalences s'impliquent mutuellement; un exemple montre qu'elles n'impliquent pas l'équivalence uniforme.
C. Pauc (le Cap).

van der Waag, E. J. Équivalence d'arc et de corde. II. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 404-417 (1951).

(Suite de l'analyse précédente.) (3, 3) Une condition nécessaire et suffisante pour que dans un voisinage de O la courbe k soit rectifiable et présente en O l'équivalence uniforme d'arc et de corde, est l'existence d'une représentation paramétrique $p(t)$ telle que $p(0) = O$ et

$$\lim_{t', t'' \rightarrow 0} \frac{|p(t') - p(t'')|}{|t' - t''|} = l \neq 0.$$

(3, 4) La proposition correspondante dans le cas d'équivalence non uniforme n'est pas valide. (3, 6) Pour que la courbe plane rectifiable k admette en O l'équivalence ordinaire d'arc et de corde, il faut et il suffit qu'on puisse trouver sur un intervalle $I(-\delta, +\delta)$ une représentation $x = \xi(t)$, $y = \eta(t)$ de k telle que $\xi(0) = \eta(0) = O$ et

$$1) \quad \lim (\xi^2(t) + \eta^2(t))/t^2 = l > 0,$$

2) les nombres dérivés de ξ et η sont bornés, 3) $\xi'(t)$ et $\eta'(t)$ existent presque partout sur I tandis que $\lim (\xi^2(t) + \eta^2(t)) = l$. Un exemple est donné montrant que la proposition devient inexacte si 2) est abandonnée. (3, 7-8) Une courbe k est dite "aplatie" en O si les triangles voisins de O sur k ont un angle tendant vers zéro. Sur une courbe rectifiable l'équivalence uniforme d'arc et de corde en O implique l'aplatissement en O . (3, 9) Un exemple est donné montrant que la condition d'aplatissement n'implique pas l'existence d'un voisinage rectifiable de O sur k ni en cas de rectifiabilité, l'équivalence ordinaire d'arc et de corde en O . (3, 10) Si k présente en O l'équivalence contrastante d'arc et de corde, si O est intérieur à k et si les demi-tangentes en O existent, ces dernières sont opposées. Sans la condition d'existence des demi-tangentes, la tangente ordinaire peut ne pas exister. [Remarques du rapporteur. (i) Le rapporteur comprend l'exemple de (3, 6) si $t = 3 - 3p$ et si $Q_1 P_2, Q_2 P_3, \dots$ repré-

sentent des segments linéaires. (ii) Dans la proposition de (3, 6), 2) peut être remplacée par "ξ et η sont absolument continues". (iii) La plupart des résultats de (3, 7-10) sont contenus dans un mémoire de G. Choquet [Bull. Soc. Math. France 71, 112-192 (1944); *ces Rev.* 7, 36; cf. Ch. IV, §§1, 6, et surtout Ch. V, §§4 (Th. III) et 6 (Th. V).]

C. Pauc (le Cap).

Strubecker, Karl. Erlanger Programm und Differentialgeometrie. Math.-Phys. Semesterber. 2, 263-278 (1952).

Müller, Hans Robert. Zur Geometrie der dreigliedrigen Bewegungsvorgänge. Monatsh. Math. 55, 330-339 (1951).

In the euclidean 3-space R' another 3-space R is moving. Its motion is described by means of a reference frame $P = \{p; r_1, r_2, r_3\}$ which is mobile with respect to both R and R' [$r_i x_i = \delta_{ij}$]. The motions of R and P depend on three parameters u_1, u_2, u_3 . Let $\delta[\delta']$ indicate differentials computed in R [R'] and let i, j, k denote cyclic permutations of 1, 2, 3. Then the differentials of P in R are given by

$$\delta r_i = r_j \omega_k - r_k \omega_j; \quad \delta p = r_1 \bar{\omega}_1 + r_2 \bar{\omega}_2 + r_3 \bar{\omega}_3$$

where the ω_i and $\bar{\omega}_i$ are Pfaffians in u_1, u_2, u_3 [$i=1, 2, 3$]. Corresponding formulas with Pfaffians ω'_i and $\bar{\omega}'_i$ hold in R' . Let $\psi_i = \omega'_i - \omega_i$, $\bar{\psi}_i = \bar{\omega}'_i - \bar{\omega}_i$. If $\xi = p + \sum r_i x_i$ is fixed in R , then $\delta \xi = 0$ implies $\delta' \xi = \sum r_i (\psi_i - x_i \psi_j + x_j \psi_i)$.

If the original ψ_1, ψ_2, ψ_3 are linearly independent, then it is possible to introduce a canonical reference frame for which $\bar{\psi}_i = c_i \psi_i$ [$c_i = c_i(u_1, u_2, u_3)$, $i=1, 2, 3$]. The quadric with the center p and the axis vectors $c_i x_i$ is uniquely determined by the condition that P is canonical.

Let $\xi[\mathcal{E}]$ be a point [plane] in R . In R' , $\xi[\mathcal{E}]$ describes a three-dimensional manifold of points [planes]. It has a volume element dV [a planar density dE in the sense of integral geometry]. The computation of dV and of dE leads to characterizations of the vectors r_i of the canonical P . Thus e.g., all the $\xi \subset R$ with the same dV lie on a quadric whose principal axes are parallel to these vectors. The quadric of those $\xi \subset R$ for which $dV=0$ ["flächenläufige Punkte"] is connected with still another characterization which is said to generalize one given by Beth [cf. K. Federhofer, Graphische Kinematik und Kinetostatik, Springer, Berlin, 1932, p. 60].

P. Scherk (Los Angeles, Calif.).

Müller, Hans Robert. Über zwangsläufige Bewegungsvorgänge. Collectanea Math. 3, 3-10 (1950).

Nach Study's Übertragungsprinzip der Liniengeometrie können die gerichteten Geraden des dreidimensionalen Raumes die dualen Punkte der Einheitskugel eindeutig zugeordnet werden, wobei die dualen Drehungen der Kugel die Bewegungen des Raumes entsprechen. In dieser Weise behandelt der Verf. die zwangsläufigen Bewegungen des Raumes mittels dualer Zahlen; die Rechnung wird noch vereinfacht indem er nicht nur die Gangkugel und die Rastkugel betrachtet, sondern einem Verfahren van der Woude's folgend eine geeignet gewählte dritte Kugel, welche sich bezüglich der beiden andern bewegt. Die Betrachtung wird angewendet auf die Bewegung einer Geraden im Raum und es wird z.B. den Satz bewiesen: die Geraden des Raumes, welche augenblicklich Regelflächen gegebenen Dralls (Verteilungsparameter) beschreiben, bilden einen quadratischen Komplex.

O. Bottema (Delft).

Lauffer, R. Analytische Kurvenpaare auf einer Fläche 2.

O. Arch. Math. 2 (1949-1950), 461-465 (1951).

Let $X^{(0)}, X^{(1)}, X^{(2)}, X^{(3)}$ denote the coordinates of the point X in complex projective 3-space. Given the quadric $\Omega = \sum a_{ab} X^{(a)} X^{(b)} = 0$ [$a_{aa} = a_{ii}$; $|a_{ab}| = 1$]. Put

$$(A|B) = \sum a_{ab} A^{(a)} B^{(b)}.$$

Let $UC\Omega$ and $VC\Omega$ depend analytically on the same complex parameter t . Thus $(U|U) = (V|V) = 0$. Dots denote differentiation with respect to t . Suppose $(U|V)$, $(\dot{U}|\dot{U})$ and $(\dot{V}|\dot{V})$ do not vanish. Then the coordinates of U and V can be normed and t can be chosen such that

$$(U|V) = (\dot{U}|\dot{U}) = (\dot{V}|\dot{V}) = 1.$$

Put $\Phi = (U|\dot{V})$, $\Psi = (\dot{U}|V)$, and $D_3 = (\dot{U}|\dot{V}) - (\dot{U}|\dot{V})$. Then every elementary invariant of the pair of curves U, V is a rational function of Φ, Ψ, D_3 , and their derivatives. The author also gives geometrical interpretations of these invariants.

P. Scherk (Los Angeles, Calif.).

Wunderlich, Walter. Über die Torsen, deren Erzeugenden zwei Kugeln berühren. Soc. Sci. Fenn. Comment. Phys.-Math. 14, no. 10, 16 pp. (1949).

Wunderlich, Walter. Über die Nyströmsche Strahlkongruenz und die geodätischen Linien der Flächen 2. Grades. Soc. Sci. Fenn. Comment. Phys.-Math. 15, no. 11, 8 pp. (1950).

[Theorem π of the first, respectively second, paper will be quoted as Th. I, π , respectively Th. II, π]. Bricard proved Th. II, 7: If two common tangents of two quadrics Φ and Ψ are intersected with the quadrics of the pencil through Φ and Ψ , then a projectivity between these tangents is induced [Nouvelles Ann. Math. (4) 7(67), 21-25 (1908)]. The author gives a synthetic proof of a generalization [Th. II, 8] of Th. II, 7. If the dual [Th. II, 9] of Th. II, 7 is specialized to pencils containing the absolute conic, then a metric theorem on the common tangents of confocal quadrics is obtained [cf. Th. II, 4]. It leads in its turn to a theorem by Bricard on the geodesics of a quadric [cf. Th. II, 6].

The common tangents of two spheres Φ and Ψ form a congruence studied by Nyström [Soc. Sci. Fenn. Comment. Phys.-Math. 7, no. 3 (1933); 9, no. 7 (1936)]. Let F_1 and F_2 denote the centers of the null-spheres of the pencil of spheres through Φ and Ψ [$F_1 \neq F_2$]. Suppose the common tangent t of Φ and Ψ touches Φ at P and let X be one of the points of intersection of t with a given third sphere of our pencil. A suitable duality leads from Th. II, 4 to the author's result that the angles of the triangles PF_1X are independent of the choice of t [$i=1, 2$; cf. Th. II, 1; actually, this result is the starting point of both papers].

Nyström proved that his congruence contains the tangent surfaces of two one-parametric families of curves k , one of them lying on Φ the other one on Ψ . By Th. II, 1, such a curve k intersects the straight lines through F_1 and those through F_2 at constant angles [Th. I, 1]. The same duality that leads from Th. II, 1 to Th. II, 4 transforms k into a geodesic on a special quadric [Th. I, 4; cf. also Th. II, 6]. An inversion with the center F_1 which maps Φ onto itself transforms a curve k on Φ into a congruent curve [Th. I, 2]. If the cone through k and any point of the axis F_1F_2 is developed into a plane, then k is mapped on a cyclic curve [Th. I, 3]. The last two sections of the first paper deal with the limit case $F_1 = F_2$.

P. Scherk (Los Angeles, Calif.).

Marussi, Antonio. Determinazione a priori del modulo di deformazione lineare nella rappresentazione conforme di Gauss. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) 11, 198-201 (1951).

It is well known that if a surface S with Gaussian curvature K is mapped conformally on a plane, then μ , the logarithm of the modulus of deformation, must satisfy $\Delta_2 \mu = K$. The author takes S to be a surface of rotation referred to a longitude coordinate λ and an isometric latitude coordinate ϕ , the condition on μ then becoming

$$\frac{\partial^2 \mu}{\partial \phi^2} + \frac{\partial^2 \mu}{\partial \lambda^2} = \frac{N \cos^2 \phi}{\rho} = R,$$

where ϕ is the latitude coordinate, ρ the radius of curvature of the meridian, and N the distance along the normal to the axis of rotation. Assuming that $\mu=0$ and $d\mu/dn=0$ along the meridian $\lambda=0$, the author works out the series

$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{\lambda^{2r}}{(2r)!} \frac{d^{2(r-1)} R}{d\phi^{2(r-1)}}$$

as a solution for μ . *A. Schwartz* (New York, N. Y.).

Bilharz, H. Integralumformungen und alternierende Differentialformen. *Math.-Phys. Semesterber.* 2, 238-250 (1952). Expository paper.

Cherep, Rebeca. Affine invariants of certain triples of curves in space. *Gaz. Mat., Lisboa* 12, no. 50, 35-38 (1951). (Spanish)

The author studies triples of curvilinear elements of the second order in Euclidean 3-space, each element determining a center point, a tangent line, and an osculating plane. Nine affine invariants are found in the general case, three in the special case where the curves have an ordinary point in common, and one in the still more special case where the curves have a common tangent at an ordinary point. Metric and affine interpretations for the invariants are given in the three cases. The six projective invariants determined by P. Buzano in the general case [*Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat.* 81-82, 109-113 (1948); these *Rev.* 10, 144] are included among those obtained here. The invariant obtained in the third case is simply related to those studied by L. A. Santaló [*Duke Math. J.* 14, 559-574 (1947); these *Rev.* 9, 201]. *A. Schwartz* (New York, N. Y.).

Beckert, Herbert. Über die Verbiegung von Flächenstücken positiver Krümmung und einige Bemerkungen zum Verhalten der Lösungen partieller Differentialgleichungen im Übergangsgebiet. *Math. Nachr.* 5, 123-128 (1951).

The problem of bending a bounded, simply connected, regular surface \mathfrak{F}_0 which has positive Gaussian curvature K_0 is formulated as a perturbation boundary problem for a certain elliptic system of quasi-linear partial differential equations. It is assumed that the first fundamental quantities E_0, F_0, G_0 pertaining to \mathfrak{F}_0 are of class $C_{2+\lambda}$ (i.e., have continuous second partial derivatives which satisfy a Hölder condition with exponent λ , $0 < \lambda < 1$), and the second fundamental quantities D_0, D'_0, D''_0 of class $C_{1+\lambda}$, in the closure of the parameter domain E , which is taken to be the unit disk in the uv -plane. Then D_0 and D'_0 comprise a solution of an elliptic system of quasi-linear equations which will be referred to as (A). D_0 and D'_0 are regarded as being further characterized by a linear boundary relation

$\alpha_0 D + \beta_0 D' = \gamma_0$, where γ_0 is a function of arc length of class $C_{1+\lambda}$ defined on the boundary S of E , and α_0, β_0 are functions of u, v of class $C_{1+\lambda}$ on $E+S$ not simultaneously vanishing in that region. The bending problem discussed is that of finding a new surface \mathfrak{F} , for which the first fundamental quantities are unchanged, while the second fundamental quantities D, D' (D'' being determined from these), which must also solve the elliptic system (A), satisfy a perturbed boundary condition

$$[\alpha_0 + \epsilon(\alpha - \alpha_0)]D + [\beta_0 + \epsilon(\beta - \beta_0)]D' = \gamma_0 + \epsilon(\gamma - \gamma_0)$$

on S . The functions α, β are assumed to be of class $C_{1+\lambda}$ in $E+S$ and not to vanish simultaneously in that region and γ is supposed to be of class $C_{1+\lambda}$ as a function of arc length on S .

A. Douglis (New York, N. Y.).

Backes, F. Sur les congruences conjuguées à une surface. *Acad. Roy. Belgique. Bull. Cl. Sci.* (5) 37, 879-887 (1951).

The purpose of the author is to determine geometrically all congruences conjugate to a given surface. The construction of such congruences is accomplished in the following manner. Let S be a surface and P a generic point of S . Through the circle of intersection of the tangent plane to S at P consider a sphere Σ depending on two parameters. The line joining the characteristic points of Σ generates the most general congruence conjugate to S . [See also Grove, *Amer. J. Math.* 69, 59-69 (1947); these *Rev.* 8, 487.]

V. G. Grove (East Lansing, Mich.).

Terracini, Alessandro. Le congruenze W . *Rend. Sem. Mat. Fis. Milano* 21 (1950), 1-13 (1951).

L'auteur, prenant comme thème la théorie des congruences W , compare les méthodes de géométrie projective employées par Fubini et Tzitzeica. Fubini étudie les congruences W en opérant directement, dans l'espace ordinaire, sur les deux nappes focales, et cela au moyen de formules parfois assez compliquées, introduisant des fonctions auxiliaires étrangères au problème et masquant souvent la véritable réalité géométrique. Tzitzeica, au contraire, ramène l'étude en question à un problème de géométrie réglée, pour lequel il utilise la représentation de la droite sur l'hyperquadrique de Klein de l'espace à cinq dimensions. Ce procédé, très élégant, a le défaut d'être trop géométrique, et le manque de formules pour les deux nappes focales n'est pas sans présenter de sérieux inconvénients. L'auteur procède à une comparaison très instructive entre les deux points de vue, et une introduction opportune de la transformation de Ribaucour des congruences (quadratiques) de la quadrique de Klein lui permet de donner des significations remarquables aux fonctions auxiliaires de Fubini, et d'attirer l'attention sur un intéressant rapprochement entre les expositions, d'après Fubini et Tzitzeica, du théorème de permutabilité de Bianchi. Il termine par une description autonome des systèmes (de Bianchi) de ∞^1 surfaces transformées asymptotiques de deux surfaces fixes, qui interviennent dans le théorème de permutabilité (inversion de ce théorème), et signale un rapprochement avec un autre théorème d'inversion dû à Marcus.

P. Vincensini (Marseille).

Finikov, S. P. W -Systems. *Mat. Sbornik N.S.* 29(71), 349-370 (1951). (Russian)

In the theory of stratifiable surfaces appear W -systems which contain a two-parametric family of W -congruences such that the focal surfaces form only two one-parametric families of surfaces, connected in pairs to the congruences of the system. The rays of these congruences establish on

all surfaces one and the same so-called "rigid" correspondence of points which preserves the asymptotic lines. An example of such a W -system without such a correspondence was given by the author [Izvestiya Akad. Nauk SSSR. Ser. Mat. 9, 79-112 (1945); these Rev. 7, 32]. In this paper the author studies with the aid of exterior forms, as in his "Theory of congruences" [Moscow-Leningrad, 1950; these Rev. 12, 744], two-dimensional W -systems with two one-parametric families of surfaces, here with three-parametric families of W -congruences. The investigation is then narrowed down to all two-dimensional W -systems with rigid correspondence of lines. The surfaces which belong to such systems (focal surfaces of the W -congruences) are characterized by the property that on them the asymptotic lines of one family belong to a linear complex. The different asymptotic lines of one surface belong to different complexes, but between the asymptotic lines of all surfaces of the system exists a rigid correspondence; the corresponding asymptotic lines of one family on different surfaces belong to one and only one linear complex. The congruences of the system transfer asymptotic lines of one selected family on one focal surface into the asymptotic lines of another surface belonging to the same linear complex as the original lines.

D. J. Struik (Cambridge, Mass.).

Godeaux, Lucien. Sur les surfaces associées à une suite de Laplace terminée. Colloque de Géométrie Différentielle, Louvain, 1951, pp. 191-203. Georges Thone, Liège; Masson & Cie. Paris, 1951. 350 Belgian francs; 2450 French francs.

Let (x) be a surface having a terminating sequence L of Laplace. If the sequence L terminates to a point according to the case of Laplace and if the locus of this point does not belong to a hyperplane, then the sequence L also terminates according to the case of Goursat [this theorem is due to Bompiani, Rend. Circ. Mat. Palermo 34, 383-407 (1912)], and conversely. In this case the number of the points of the sequence L is even, and therefore can be assumed to be $2n+4$. For $n=0$, the surface (x) is ruled, and for $n=1$, the asymptotic curves of one family of the surface (x) belong to linear complexes. For $n=2$, the surface (x) has the following property: If u, v are the asymptotic curves of the surface, the tangents to the curves v at all points of a curve u belong to a linear complex. If the surface (x) is further a focal surface of a W -congruence, then the other focal surface of the congruence has a terminating sequence of Laplace of $2n+2, 2n+4$ or $2n+6$ terms. Thus, a ruled surface and a surface for which $n=1$ can be the focal surfaces of a W -congruence. This property was obtained by C. Segre and has been used by Terracini [see Fubini and Čech, Geometria proiettiva differenziale, v. 2, Zanichelli, Bologna, 1927, Appendix IV] in determining the surfaces for which $n=1$. Similarly, a surface for which $n=1$ and a surface for which $n=2$ can be the focal surfaces of a W -congruence, and this property may probably be used in determining the last surface.

C. C. Hsiung (Cambridge, Mass.).

Rozet, O. Sur les suites de Laplace de période six. Bull. Soc. Roy. Sci. Liège 20, 471-477 (1951).

Based on the methods and notations of Wilczynski [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 4° (2) 3, no. 5 (1911)] and the author [same Bull. 10, 138-167 (1941); these Rev. 7, 32] an analytic study is made of sequences of Laplace of period six. Various special cases are considered. [See also Rozet, Bull. Soc. Roy. Sci. Liège 19, 243-249, 335-342 (1950); these Rev. 12, 745.]

V. G. Grove.

Thybaud, Alexandre, et Robert, Paul. Sur les surfaces engendrées par les cercles d'une congruence paratactique. I. C. R. Acad. Sci. Paris 233, 775-777 (1951).

Thybaud, Alexandre, et Robert, Paul. Sur les surfaces engendrées par les cercles d'une congruence paratactique. II. C. R. Acad. Sci. Paris 233, 842-844 (1951).

These two articles are reports of some results on studies made of surfaces generated by the circles of a paratactic congruence: Denote the congruence by K , and by (C) the circles of K , by Σ the surfaces generated by (C) , and by L the locus of the centers of (C) . The focal sheets of K are curves G and a line g . The locus L is located in the central plane and G on the central sphere. Among the surfaces Σ are the cyclides of Dupin. Among the results reported may be cited the following. Two surfaces Σ, Σ' intersect in circles (C) ; the angle between these surfaces is constant along their common circle, and is equal to the angle between the curves G, G' at their common intersection. A surface Σ having a circle as a line of curvature is a cyclide of Dupin. The properties of Σ may be deduced from those of L . The lines of curvature on Σ may be found by quadratures, and each circle (C) cuts all of the lines of curvature of one system at a constant angle. If a line of curvature of Σ is spherical, all of the lines of curvature belonging to the same system are also spherical, and all of the sustaining spheres are orthogonal to a fixed circle (C_0) of K . The determination of isothermic surfaces Σ is equivalent to the determination of spherical curves of constant torsion.

V. G. Grove.

Hlavatý, Václav. Spinor space and line geometry. II. J. Rational Mech. Anal. 1, 321-339 (1952).

In the first part of this paper [Canadian J. Math. 3, 442-459 (1951); these Rev. 13, 687] the ideal space L_3 of the centered vector space R_4 of relativity was mapped onto the spinor space considered as a projective space S_3 with a congruence K and a linear complex Γ . These line configurations are now used as "absolute" configurations in S_3 and in this S_3 the spinor lines are considered as elements. A "line metric" is established in §§4, 5. In the §§6, 7 use is made of the biaxial involution in S_3 introduced in the first paper for a geometric interpretation of Dirac equations. This involution maps a reflexion in L_3 connected with Schrödinger's relativistic wave equation.

J. A. Schouten (Epe).

Santaló, L. A. On permanent vector-varieties in n dimensions. Portugaliae Math. 10, 125-127 (1951).

The author obtains necessary and sufficient conditions for the permanence of r -dimensional vector-varieties in Euclidean n -space, thus generalising the result of the reviewer for $r=1$ [Proc. Amer. Math. Soc. 2, 370-372 (1951); these Rev. 12, 868; for $n=3, r=1$ see also Prim and Truesdell, ibid. 1, 32-34 (1950); these Rev. 11, 696]. With Latin suffixes ranging 1, ..., n and Greek 1, ..., r , let $c_a^i(x, t)$ be a set of r moving vector fields, x^i being the coordinates and t a parameter (time); let $v^i(x, t)$ be another vector field, regarded as the velocity of a fluid filling the n -space. Consider a subspace V , moving with the fluid; it is defined to be a vector-variety if at each point in it the vectors c_a^i are tangent to it. The problem is to find conditions on c_a^i and v^i so that there exist vector-varieties which are permanent, i.e. have the definitive property for all t . The conditions are stated in terms of the vanishing of certain $(r+1)$ -vectors,

the $(r+1)$ -vector formed from any $r+1$ ordinary vectors $T_1^i, T_2^i, \dots, T_{r+1}^i$ being defined by the determinantal formula

$$T^{i_1 i_2 \dots i_{r+1}} = \begin{vmatrix} T_1^{i_1} & \dots & T_{r+1}^{i_1} \\ \vdots & \ddots & \vdots \\ T_1^{i_{r+1}} & \dots & T_{r+1}^{i_{r+1}} \end{vmatrix} = [T_1, T_2, \dots, T_{r+1}].$$

Necessary and sufficient conditions for permanence are found to be $Z_r = 0$, where Z_r is the $(r+1)$ -vector

$$Z_r = \left[v_{,k} c_r^k - c_{r,k} v^k - \frac{\partial c_r}{\partial t}, c_1, c_2, \dots, c_r \right].$$

J. L. Synge (Dublin).

Vaona, Guido. Sulla trasformazione linearizzante di una corrispondenza puntuale fra spazi lineari. Boll. Un. Mat. Ital. (3) 6, 293-299 (1951).

Using some notions of Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 145-151 (1949); these Rev. 10, 738], Villa [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 1-7 (1943), p. 5; these Rev. 8, 219] and others, the author gives a new geometric definition to the linearized transformation, introduced by Čech [Časopis Pěst. Mat. Fys. 74, 32-48 (1950); 75, 123-126, 137-158 (1950); these Rev. 12, 534; 13, 158], between two linear spaces S_r of $r(\geq 2)$ dimensions. It is showed that the problem, proposed by Villa [Atti del III Congresso dell'Un. Mat. Ital., Pisa, 1948, pp. 157-159, p. 159, Perrella, Rome, 1951], of determining point transformations between two linear spaces S_r of $r(>2)$ dimensions, which have a cone V_{r-1} of characteristic directions at each point, is equivalent to the one which has been solved by Čech in the papers cited above. Finally, another problem expressed in terms of the notion of the linearized transformation is also showed to be equivalent to the one of determining point transformations between two S_r which have a S_{r-1} of characteristic directions at each point.

C. C. Hsiung (Cambridge, Mass.).

Bell, P. O. A theorem on conjugate nets in projective hyperspace. Proc. Amer. Math. Soc. 3, 300-302 (1952).

As a generalization of a theorem of the reviewer [Trans. Amer. Math. Soc. 70, 312-322 (1951); these Rev. 13, 276], the following theorem is proved. Let N_n be a conjugate net in a linear space S_n of $n(\geq 3)$ dimensions, and let M, \bar{M} be two points respectively on the two tangents at a point x of the net N_n . If the tangent plane at M (\bar{M}) of the net N_M ($N_{\bar{M}}$) described by M (\bar{M}) passes through \bar{M} (M), then N_M and $N_{\bar{M}}$ are conjugate nets and each one of them is a Laplace transformed net of the other one. For $n=3$, replacing the conjugate net N_n by an asymptotic net yields a generalization of a theorem of Su [Univ. Nac. Tucumán. Revista A. 5, 363-373 (1946); these Rev. 8, 602].

C. C. Hsiung (Cambridge, Mass.).

Kanitani, Joyo. Sur l'espace à connexion projective majorante. III. Jap. J. Math. 20, 45-54 (1950).

[For parts I, II see same J. 19, no. 3, 343-361 (1947); no. 4, 395-403 (1948); these Rev. 11, 54; 12, 360.] In paper II the author showed that an n -dimensional space R_n with a symmetric projective connection can be imbedded in a projective space S_N of dimensions $N = \frac{1}{2}n(n+1) + n - 1$ in such a way that this space R_n becomes a variety V_n generated by ∞^{n-1} straight lines. This paper is devoted to a study of conditions under which the dimensionality of the space S_N can be diminished. The most general conditions are contained in the following result. If the space R_n admits ∞^{n-k}

totally geodesic varieties (L_k) of k dimensions and ∞^{n-k-k} directrix varieties (R_{k+m}) of $k+m$ dimensions of the system (L_k), and if the projective transformation associated to an infinitely small closed curve, starting and ending at a point x^i , on a variety R_{k+m} always leaves invariant all the points in the space S_k which is the locus of the developments of all curves issuing from the point x^i and lying on the variety L_k passing through the point x^i , then the space R_n can be imbedded in a projective space S_N ($N = \frac{1}{2}n(n+1) + n - \frac{1}{2}k(k+1) - km$) in such a way that this space R_n becomes a variety V_n generated by ∞^{n-k-m} varieties of $k+m$ dimensions each of which is an envelope of ∞^m projective spaces of $k+m$ dimensions, and that at any point of V_n the osculating planes of all curves on V_n through this point determine the space S_N .

C. C. Hsiung (Cambridge, Mass.).

Bartsch, Helmut. Hyperflächengewebe des n -dimensionalen Raumes. Ann. Mat. Pura Appl. (4) 32, 249-269 (1951).

This is the study of the differential invariants of webs of hypersurfaces in an n -dimensional space R_n , in the sense of Blaschke [Blaschke und Bol, Geometrie der Gewebe . . . , Springer, Berlin, 1938]. The method adopted is that of Elie Cartan of differential forms. Most of the paper is concerned with $(n+1)$ -webs, for which a complete system of invariants is determined and the results geometrically interpreted as defining an affine connection in the space. [Note of the reviewer: It is of some importance to remark that the group of holonomy of this affine connection is a certain proper subgroup of the affine group.] The author then applies his analytical theory to some geometrical problems, namely, conditions that the web is equivalent to pencils of hyperplanes, properties of the webs of intersection, conditions for certain configurations to be closed, etc. A last section is devoted to prove a theorem on $(n+2)$ -webs of hypersurfaces in R_n .

S. Chern (Chicago, Ill.).

Pratelli, Aldo M. Lavoro e flusso dei tensori emisimmetrici. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 13(82), 12-28 (1949).

The author generalizes the familiar notions of the work and flux of a vector. Suppose that an m -dimensional submanifold of an n -dimensional Riemannian manifold has equations

$$x^i = x^i(\xi_\alpha) \quad (i=1, 2, \dots, n; \alpha=1, 2, \dots, m).$$

Let σ be a region in this submanifold, and let $V_{i_1 \dots i_m}$ be an alternating (i.e. skew symmetric) tensor-field defined therein. Then the author defines the element of work of the tensor to be

$$d^*L = V_{i_1 \dots i_m} \frac{\partial x^{i_1}}{\partial \xi_1} \dots \frac{\partial x^{i_m}}{\partial \xi_m} d\xi_1 \dots d\xi_m,$$

and the total work over σ to be $\int_\sigma d^*L$. If $U^{i_1 \dots i_m}$ is an alternating contravariant tensor-field, then its flux across σ is defined to be

$$\Phi^{i_1 \dots i_m} = \frac{1}{(n-m)!} \int_\sigma U^{i_1 \dots i_m} N_{k_1 \dots k_{n-m}} d\sigma,$$

where $s = r - n + m$ and $N_{k_1 \dots k_{n-m}}$ is the multivector normal to σ . The work of an alternating tensor of order m over σ is equal to the flux of the conjugate (dual) tensor. The author also defines the generalized div, curl and grad of an alternating tensor, and discusses the generalized Stokes's theorem. A passing reference is made to the theory of

harmonic forms, the remark being made that a harmonic alternating tensor is one that is both solenoidal and irrotational.

H. S. Ruse (Leeds).

Castoldi, Luigi. Tensori simmetrici in varietà riemanniane tridimensionali con metrica definita positiva, e loro rappresentazione mediante vettori. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 688-694 (1950).

If $a_{ij\dots k}$ is a covariant tensor, symmetric in all its n suffixes, at a point P of a Riemannian 3-space of positive-definite metric, then, by a theorem of Sylvester, the n -form $F^{(n)} = a_{ij\dots k} \xi^i \xi^j \dots \xi^k$ is uniquely expressible in terms of a symmetric $(n-2)$ -form $F^{(n-2)}$ by the formula

$$F^{(n)} = A^{(n)} u_i v_j \dots w_k \xi^i \xi^j \dots \xi^k + g_{ij} \xi^i \xi^j F^{(n-2)},$$

where u_i, v_i, \dots, w_i are vectors, $A^{(n)}$ is a scalar, and g_{ij} is the fundamental tensor at P . Repeated application of this formula to $F^{(n-2)}, F^{(n-4)}, \dots$ leads to the expression of $F^{(n)}$, and hence of $a_{ij\dots k}$, in terms of vectors and scalars. The case $n=2$ admits of a simple geometrical interpretation. There is no corresponding formula in a space of dimensions other than 3.

H. S. Ruse (Leeds).

Aragnot, André. Géométrie globale des espaces d'éléments linéaires à connexion euclidienne. C. R. Acad. Sci. Paris 234, 1426-1428 (1952).

"Cette note a pour but d'étendre au cas des connexions euclidiennes d'éléments linéaires la formule intégrale de Chern donnant la caractéristique d'Euler-Poincaré d'une variété plongée dans un espace de Riemann." (From the author's summary.)

S. Chern (Chicago, Ill.).

Hlavatý, Václav. Intrinsic deformation theory of subspaces in a Riemann space. J. Rational Mech. Anal. 1, 49-72 (1952).

In a Riemannian space V_n consider a set of subspaces V_m ($1 \leq m < n$) such that in a certain region R of V_n there passes one and only one V_m . A family (V_m) of such V_m is represented parametrically by

$$(1) \quad \xi^r = \phi^r(\tau^1, \dots, \tau^m; \nu^{m+1}, \dots, \nu^n).$$

If we denote by Y the space obtained by identifying all points of (V_m) which have the same value of the parameter τ , the equation (1) may be represented symbolically by

$$(2) \quad R = \phi(Y)$$

and this equation represents a mapping of Y into R . The family (V_m) is investigated with respect to infinitesimal deformations

$$(3) \quad \delta \xi^r = \xi^r + \epsilon V^r(\xi).$$

This paper is a sequel to another paper on the same subject [Proc. Amer. Math. Soc. 1, 600-617 (1950); these Rev. 12, 358]. The treatment in this paper differs from that in the preceding paper in that some of the results in the first paper depend on the mapping (2). The method used here is intrinsic, and all results are independent of the mapping. The author states and proves certain theorems concerning the preservation of (V_m) under the infinitesimal deformation (3).

E. T. Davies (Southampton).

Ôtsuki, Tominosuke. On the spaces with normal conformal connexions and some imbedding problem of Riemannian spaces. II. Tôhoku Math. J. (2) 2, 220-274 (1951).

In a previous paper [same J. (2) 1, 194-224 (1950); these Rev. 11, 742] the author studied spaces with a normal con-

formal connection whose group of holonomy fixes a point or a hypersphere and the question of imbedding a Riemannian space as a hypersurface of such a space. This investigation is continued in this paper. The imbedding problem leads to complicated differential systems, which are analysed in detail.

S. Chern (Chicago, Ill.).

Ryžkov, V. V. An imbedding theorem for Riemannian geometries of higher order. Doklady Akad. Nauk SSSR (N.S.) 75, 503-506 (1950). (Russian)

Here relations between n -dimensional surfaces $\bar{x} = \bar{x}(u_i)$, $\bar{y} = \bar{y}(u_i)$, $i = 1, \dots, n$ in euclidean R_N are discussed, where isometry, that is, equality of the forms $\omega_1 = d\bar{x}^2$ and $\omega_1' = d\bar{y}^2$, is replaced by conditions

$$\frac{\partial \alpha_1 + \partial \alpha_2 + \dots + \partial \alpha_n}{\partial u_1^{\alpha_1} \partial u_2^{\alpha_2} \dots \partial u_n^{\alpha_n}} = \frac{\partial \beta_1 + \partial \beta_2 + \dots + \partial \beta_n}{\partial u_1^{\beta_1} \partial u_2^{\beta_2} \dots \partial u_n^{\beta_n}},$$

$$= \frac{\partial \alpha_1 + \partial \alpha_2 + \dots + \partial \alpha_n}{\partial u_1^{\alpha_1} \partial u_2^{\alpha_2} \dots \partial u_n^{\alpha_n}} = \frac{\partial \beta_1 + \partial \beta_2 + \dots + \partial \beta_n}{\partial u_1^{\beta_1} \partial u_2^{\beta_2} \dots \partial u_n^{\beta_n}},$$

$$1 \leq \alpha_1 + \dots + \alpha_n \leq k, \quad 1 \leq \beta_1 + \dots + \beta_n \leq k.$$

It is sufficient for this that the forms $\omega_s = (d\bar{x})^2$, $\omega_s' = (d\bar{y})^2$ are pairwise equal for $s = 1, 2, \dots, k$. Some of the work done by Schlaefli, Janet and Cartan on the imbedding problem for $k=1$ is here generalized for arbitrary k . The result is given by the theorem that if a set of forms $\omega_s(u_i, du_i)$, $s = 1, 2, \dots, k$ is given (analytical coefficients, positive definite, order $2s$) then there exists a surface $\bar{x} = \bar{x}(u)$ with $(d\bar{x})^2 = \omega_s$, in a euclidean R_N :

$$N = \sum_{s=1}^k \frac{n(n+1) \dots (n+2s-1)}{(2s)!}.$$

In the case that for the \bar{x} all vectors

$$d\bar{x}, d^2\bar{x}, \dots, d^k\bar{x}, \delta_1 d^{k-1}\bar{x}, \dots, \delta_1^2 d^{k-2}\bar{x}$$

are linearly independent, then the \bar{x} may depend on

$$N_1 = \sum_{s=0}^{k-1} (k-s) \frac{n(n+1) \dots (n+2s)}{(2s+1)!}$$

functions of $n-1$ variables. (Here δ_1 is differentiation with u_1 constant.)

D. J. Struik (Cambridge, Mass.).

Dubnov, Ya. S. The centro-affine geometry of curves in the plane. Trudy Sem. Vektor. Tenzor. Analizu 8, 106-127 (1950). (Russian)

Centro-affine geometry is considered as projective geometry with a hyperplane ω and a point O , not in ω , fixed. Two points not in ω determine a vector \bar{a} , two hyperplanes not through O a so-called "doublet" \bar{a} . For the case of the plane the vector algebra of these entities is developed, with their geometrical meaning, and this is also done for the equi-centro-affine case. When a curve $\bar{r}(u)$ is given, then the doublet $\bar{r}(u)$ indicates the tangent lines if $\bar{r} = 1$; $\bar{r}_1 = d\bar{r}/du$ is parallel to \bar{r} ; $\bar{r}_1 = d\bar{r}/du$ is parallel to the \bar{r} of the tangent point. In this symbolism the concepts of affine differential geometry of curves are derived. The transition to affine, centro-projective and projective geometry is derived. The cube $d\sigma^3$ of the natural projective parameter is shown to be $d\kappa ds^3$, where κ is the equi-affine curvature and ds the natural equi-affine parameter.

D. J. Struik (Cambridge, Mass.).

Dubnov, Ya. S., and Skrydlov, V. N. The centro-affine theory of surfaces. Trudy Sem. Vektor. Tenzor. Analizu 8, 128-143 (1950). (Russian)

The vector algorithm for centro-affine spaces, derived in the paper reviewed above is here applied to surface theory of space of three dimensions. The equiaffine space is specially studied. Formulas are found for the net of asymptotic lines, the invariant of Tzitzeica $T = \frac{1}{2} \epsilon^{\alpha\beta\gamma} R_{\alpha\beta} R_{\gamma\delta}$ ($R_{\alpha\beta}$ the curvature tensor), and the Čebyšev tensor (the tensor which vanishes when a net is a net of Čebyšev for a given connection). The transition is made to affine centro-projective geometry. D. J. Struik (Cambridge, Mass.).

Hashimoto, Shintaro. A new proof of Liber's theorem. Kōdai Math. Sem. Rep. 1951, 118-119 (1951).

As the title indicates, this note gives a different proof of Liber's theorem [Doklady Akad. Nauk SSSR (N.S.) 66, 1045-1046 (1949); these Rev. 11, 134]: If the holonomy group of an affine space without torsion is of one parameter, and its symbol is $Xf = a_i x^i \partial f / \partial x^i$, then the rank of a_i is at most two. The proof is based on E. Cartan's theorem on pfaffian forms. M. S. Knebelman (Pullman, Wash.).

Akivis, M. A. Invariant construction of the geometry of a hypersurface of a conformal space. Doklady Akad. Nauk SSSR (N.S.) 82, 325-328 (1952). (Russian)

One of the potent methods for the study of differential geometry is that of the "repère mobile". For conformal geometry it consists of two points A_0, A_{n+1} and n spheres A_1, \dots, A_n . The scalar products of the elements of this repère satisfy a series of relations, because of which an infinitesimal displacement of the repère, ω^μ ($\mu = 0, 1, 2, \dots, n+1$) satisfies a set of linear equations. For a hypersurface A_0 is a point of it, A_n is a hypersphere tangent to the hypersurface at A_0 and A_1, A_2, \dots, A_{n-1} are hyperspheres orthogonal to A_n ; $\omega^n = 0$ is the pfaffian equation of the hypersurface. The scalar

$$(A_i A_j) = g_{ij} \quad (i, j = 1, \dots, n-1)$$

is non-singular and the angular metric $g_{ij} \omega^i \omega^j$ is then positive definite. The differential invariants of the hypersurface are then obtained as coefficients of extended pfaffians: thus $\omega^\mu = \alpha_{ij} \omega^i \omega^j$, $\tilde{d} \alpha_{ij} = \alpha_{ij} \omega^0 - g_{ij} \omega^0 + \beta_{ijk} \omega^k$ etc. where \tilde{d} is covariant differential with respect to $\omega^i - \delta_j^i \omega^0$. The main point of the paper, however, is the construction of the differential invariants of the hypersurface that have a clearer geometrical meaning. Defining $a = g^{ij} \alpha_{ij} / (n-1)$ one obtains $da = -a \omega^0 - \omega_n^0 + \beta_k \omega^k$, where $\beta_k = g^{ij} \beta_{ijk} / (n-1)$. From this one gets the tensor $a_{ij} = \alpha_{ij} - a g_{ij}$ so that

$$d\beta_k = -\beta_k \omega^0 - a \omega_{n+1}^k + \gamma_{kl} \omega^l.$$

If $\det a_{ij} \neq 0$, one can get an absolute tensor

$$c_{ij} = \gamma_{ij} + a a_{ij} - \frac{1}{2} (g_k b^k b^l - a^2) g_{ij} + b_i b_j$$

where $b^k = a^{kl} b_l$. The tensors a_{ij} and b_{ijk} are apolar to g_{ij} and these together with c_{ij} satisfy relations similar to the Gauss-Codazzi equations of metric geometry. The conformal geometry of the hypersurface is further studied by means of osculating spheres. Thus the sphere $C = a A_0 + A_n$ is the one such that the directions at points of contact of the second order satisfy $a_{ij} \omega^i \omega^j = 0$. In case a_{ij} is not degenerate the family of central hyperspheres depends on $n-1$ parameters. If the rank of a_{ij} is m , $m < n-2$, the family depends on m parameters; if $m = n-2$, there may be two cases, in one the number of parameters being $n-2$, in the other $n-1$. The other questions considered deal with the existence of

cyclides, the interesting case being that of conformal three-space. Finally the author gives a geometric meaning to the tensor c_{ij} ; thus if $c_{ij} = g^{kl} c_{klj}$, c_{ij} is called almost orthogonal if $c^k c_k = \lambda^2 \delta^k_k$. If c_{ij} is symmetric and c_{ij} is almost orthogonal, then $c_{ij} = \lambda \delta_{ij}$. In this case the hypersurface is—up to a conformal transformation—minimal, of hyperbolic, euclidean or elliptic type according as λ is $>$, $=$ or < 0 .

M. S. Knebelman (Pullman, Wash.).

Vagner, V. V. The geometry of Finaler as a theory of the field of local hypersurfaces in X_n . Trudy Sem. Vektor. Tenzor. Analizu 7, 65-166 (1949). (Russian)

The geometry of Finsler in an X_n is in this paper systematically discussed on the basis of the author's general theory of associated manifolds $X_{n+(n-1)}$, developed in 1943 and later [see e.g. same Doklady 66, 1033-1036 (1949); these Rev. 11, 134]. In §1 he discusses the basic theory of hypersurfaces and their dual form, hyperfamilies of hyperplanes in centro-affine space, which theory was developed for $n=3$ by O. Mayer [Ann. Sci. Univ. Jassy. Partie I. 21, 1-77 (1934)]. In §2 we find the contravariant and covariant vector metric in such a space; here the moduli of measuring vectors are considered as functions of their components. This theory takes the place of that of a Minkowski metric in other Finsler theories, and allows full duality. Then follows in §3 a dual theory for the parallel displacement of vectors. Here enters the notion of curvature and the possibility of introducing parallel displacement of contravariant vectors into Finsler space. The corresponding parallelism of covariant vectors allows the introduction of geodesics. Here appears as a metrical contravariant vector connection of curvature zero the connection of L. Berwald [Jber. Deutsch. Math. Verein. 34, 213-220 (1925); Math. Z. 25, 40-73 (1926)]. In §4 the theory is applied to the calculus of variations. Here the geometry of Finsler is built up on the theory of local hypersurfaces in X_n . The author then shows in §5 how the method of Cartan [Les espaces de Finsler, Actualités Sci. Ind., no. 79, Hermann, Paris, 1934] takes a clearer geometrical form by the use of a central affine geometry of hypersurfaces. In §6 this is carried out by considering the geometry of Finsler as the geometry of spaces of linear elements. For the wealth of special results and the details of the method we must refer to the paper itself.

D. J. Struik (Cambridge, Mass.).

Vagner, V. V. The geometry of space with a hyperareal metric as the theory of local hypersurfaces in a composite manifold. Trudy Sem. Vektor. Tenzor. Analizu 8, 144-196 (1950). (Russian)

This paper is based on an idea of E. Cartan [Les espaces métriques fondés sur la notion d'aire, Actualités Sci. Ind., no. 72, Hermann, Paris, 1933; reference is to a Russian translation]. An m -areal metric is introduced into an affine E_n by means of a density of weight -1 ,

$$\chi(\kappa) = \int \dots \int dx^1 \dots dx^n,$$

taken over a domain R in a coordinate system (κ) of an E_m . A space X_n with a hyperareal metric is defined as an X_n in which every tangent space has a hyperareal metric. It is possible to approach Finsler geometries in this way, using the notion of the associated $X_{n+(n-1)}$, which has been specially investigated by the author in other papers. The main task is always the determination in this associated manifold of an appropriate invariant linear displacement. There is a chapter on hypersurfaces of extremal hyperplanes and on

so-called transformations of Carathéodory, which relate two given hyperareal metrics.

D. J. Struik.

Kaganov, S. A. On the geometrical theory of a singular variational problem for $n-1$ -fold integrals. Doklady Akad. Nauk SSSR (N.S.) 75, 487-490 (1950). (Russian)

A hyperareal metric in an affine E_n is equivalent to a covariant vector metric in an associated central affine space \mathbb{E}_n of vector densities of weight $+1$. The covariant metric in \mathbb{E}_n can be regular or singular [V. V. Vagner, the two papers reviewed above]. In this paper a singular metric in E_n of class $(n-m)$ is discussed, based on $(m-1)$ -surfaces, complemented by a regular hyperstrip. The hyperstrip in E_n is given by $x^a = l^a(\eta^p)$, $y_a = l_a(\eta^p)$, $\alpha, \beta, \dots = 1, \dots, n$; $a, b, \dots = 1, \dots, m-1$, and the vectors which define the characteristic $(n-m)$ -planes of the hyperstrip by $n_p^a(\eta^p)$, $p = 1, \dots, n-m$. Then the geometry is constructed with the aid of the object

$$\mathfrak{H} = \frac{1}{(n-1)!} \mathfrak{G}^{-1}$$

$$\times (\mathfrak{G}^{a_1 \dots a_{n-1}} l_{a_1} l_{a_2} \dots l_{a_{n-1}} n_{n-1}^{a_1} \dots n_{n-1}^{a_{n-1}} (\mathfrak{G}^{a_1 \dots a_{n-1}} l_{a_1} \dots l_{a_{n-1}} n_{n-1}^{a_1} \dots n_{n-1}^{a_{n-1}})^s$$

where $\mathfrak{G} = \det |g_{ab}|$, $g_{ab} = l_a^i l_b^i$, $l_a^i = \partial x^i / \partial \eta^a$, $l_{ab} = \partial l_a^i / \partial \eta^b$, $\partial = \partial / \partial \eta$. The case where the x^a and y_a depend on ξ^p , η^p is then taken up, as well as the case where an extremal hypersurface is given imbedded in an X_{n-1} . This requires the investigation of the Pfaffian system $l_a d\xi^a = 0$, $n_a d\xi^a = 0$, where the η^p are now functions of the ξ^p .

D. J. Struik.

Vagner, V. V. The theory of a field of local hyperstrips. Trudy Sem. Vektor. Tenzor. Analizu 8, 197-272 (1950). (Russian)

The author continues the work undertaken in Doklady Akad. Nauk SSSR 66, 1033-1036 (1949); these Rev. 11, The first part of this new monograph deals with the 134. geometry of regular $(m-1)$ -dimensional hyperstrips in a central affine space E_n . Fundamental tensors, curvature tensors and invariant affine conceptions are introduced. It is shown how two arbitrary tensors g_{ab} , A_{ab} , $\det |g_{ab}| \neq 0$, $a, b, \dots = 1, \dots, m-1$, related to the tensors h_{ab}^{\pm} and h_{ab}^{\pm} and the object of connection G_{ab}^{\pm} , $p, q = m+1, \dots, n$, by a set of equations similar to those of Gauss-Codazzi-Ricci, determine a regular $(m-1)$ -dimensional hyperstrip but for non-arbitrary central-affine transformation in E_n . Questions of contact and incidence and some special cases of hyperstrips are discussed, among them those for which the curvature tensor is zero. In the second part we find a theory of regular $(m-1)$ -dimensional hyperstrips in a general X_n . The principal result is an invariant definition of a linear connection in the combined $X_{n+(m-1)}$ (formed by the tangent E_{m-1} and the local E_n). There are discussions on the singular metric of Finsler determined in an X_n by a field of local hyperstrips [see Vagner, Mat. Sbornik 21, 321-364 (1947); these Rev. 9, 379], on particular problems in the calculus of variations, non-holonomic geometries V_n^m in X_n and to scleronomic mechanical systems with a non-linear and non-holonomic connection [cf. L. Johnsen, Skr. Norske Vid. Akad. Oslo. I. 1941, no. 4; these Rev. 7, 223, 620 and the first cited paper of Vagner].

D. J. Struik (Cambridge, Mass.).

Oloničev, P. M. The general affine and central-projective theory of hyperstrips. Doklady Akad. Nauk SSSR (N.S.) 80, 165-168 (1951). (Russian)

By an $(m-1)$ -dimensional hyperstrip in a central-affine space E_n we understand an $(m-1)$ -dimensional hypersurface

$X_{m-1}: x(\eta^a)$ ($a, b, c = 1, 2, \dots, m-1$) such that with every one of its points a tangent hyperplane $\tilde{x}(\eta^a)$ is associated, following V. V. Vagner [see the preceding review]. In using the base vectors, $\{x, x_a, x_p\}$ in E_n and $\{\tilde{x}, \tilde{x}_a, \tilde{x}_p\}$ in E_n ($p, q = m+1, \dots, n$), of the associated hyperplanes introduced at a point η^a , the expressions for x_{ab} , x_{pa} give rise to

the affine connection $\overset{1}{G}_{ab}^i$ in X_{m-1} , connecting the affinor h_{ab}^{\pm} , and the affine connection $\overset{2}{G}_{ab}^i$, and analogously in E_n . After defining an $(m-1)$ -dimensional hyperstrip of second order (X_{m-1} is a quadric and \tilde{x} is parallel to an $(n-m)$ -direction), the author finds ones analogous to the affine normal of Blaschke and quadrics of Darboux in an affine or central-projective space. On making use of the general-affinely invariant w -density of weight $2/(m-1)$: $*w = \pm |g|^{1/(m-1)n}$ (n : the invariant point on the affine normal) and of the base vectors $\{x, x_a, x_p\}$, $\{\tilde{x}, \tilde{x}_a, \tilde{x}_p\}$, the fundamental differential equations for hyperstrips, following the fundamental theorem, are stated in a general-affine or central-projective space in terms of the invariant densities $*g_{ab}$, $*h_{ab}^{\pm}$, $*h_{ab}^{\pm}$, $*A_{ab}^{\pm} = \frac{1}{2}(*\overset{2}{G}_{ab}^i - *\overset{1}{G}_{ab}^i)$ and connections $*G_{ab}^i = \frac{1}{2}(*\overset{1}{G}_{ab}^i + *\overset{2}{G}_{ab}^i)$, G_{ab}^i . At the end it is remarked that the general-affine and central-projective theory of hypersurfaces considered by Vagner follows from the present theory as a special case.

A. Kawaguchi (Sapporo).

Penzov, Yu. E. The classification of geometric differential objects with two components. Doklady Akad. Nauk SSSR (N.S.) 80, 537-540 (1951). (Russian)

This is an elaboration of ideas of Vagner [same Doklady 46, 347-349 (1945); 69, 293-296 (1949); these Rev. 7, 265; 11, 461] for the case that a geometric object Ω has two components. The case of one component has already been solved by the author [ibid. 54, 563-566 (1946); these Rev. 9, 67]. Now it is shown that in an X_n , $n \geq 2$, the class of objects Ω with two components does not exceed 2. In an X_n , $n \geq 4$, there are no such objects of the first class. Every such object of the first class in X_3 is similar to a co- or contravariant object K which transforms as follows:

$$K_a = (A_a^i K_i + A_a^s) (A_1^i K_i + A_1^s)^{-1}, \\ K^a = (A_a^i K^i + A_a^s) (A_1^i K^i + A_1^s)^{-1}.$$

Every such object of the first class in X_3 is similar to one of the following objects: 1) a covariant vector density of weight one; 2) a combination of two objects K ; 3) the union of a scalar density of weight 1 and an object K ; 4) the ratio of two components of a covariant tensor to its third component.

D. J. Struik (Cambridge, Mass.).

Liber, A. E. On comitants of geometric differential objects. Doklady Akad. Nauk SSSR (N.S.) 80, 529-532 (1951). (Russian)

To every geometrical differential object Ω in space X_n belongs a subgroup \mathfrak{H} of the differential group $\mathfrak{D}^{(n,n)}$. The subgroup \mathfrak{H} is called the characteristic group of Ω . There exist differential objects Ω_1 , which are comitants of order s of Ω . An object Ω_1 is then and only then a comitant of an object Ω_2 if the characteristic group \mathfrak{H}_2 of Ω_2 is a subgroup of the group \mathfrak{H}_1 of Ω_1 [V. V. Vagner in his introduction to the Russian translation of Veblen and Whitehead, Foundations of differential geometry, 1949].

Let $\mathfrak{M}_i^{(r,n)}$, $i = 1, 2, \dots, v$, be the successive normal divisors of $\mathfrak{D}^{(r,n)}$; v is the class. Necessary and sufficient condition for the existence of a comitant of class $r \leq v$ is

$$\mathfrak{H}_i^{(r,n)} \neq \mathfrak{H}_i^{(r,n)}.$$

If Ω is of class $v > 1$, and \mathfrak{H} has no comitants of class $p \geq 1$ and $q \geq 1$, then

$$\mathfrak{H}N_{p+q-1}^{(v,n)} = \mathfrak{H}N_{p+q-1}^{(v,n)}, \quad \mathfrak{H}N_{p+q-1}^{(v,n)} = \mathfrak{H}N_{p+q-1}^{(v,n)},$$

which can be translated into Lie algebra, and leads to

$$N_{p+q-1}^{(v,n)} = [H_p H_q] + N_{p+q-1}^{(v,n)} CH + N_{p+q-1}^{(v,n)}$$

where N, H are the algebras belonging to $\mathfrak{H}, \mathfrak{H}$, and we have used the identities $[N_{p+q-1}^{(v,n)} N_{p+q-1}^{(v,n)}] = N_{p+q-1}^{(v,n)}$, $N_{p+q-1}^{(v,n)} = 0$ for $s \geq v$.

From these equations some conclusions are drawn: Let $p+q=v+1$, then Ω of class v ($v > 1$) has comitants of class p or of class q . If $v-1 = m_1(p_1-1) + \dots + m_s(p_s-1)$, then an Ω of class v has comitants, the class of which is equal to one of the numbers p_1, p_2, \dots, p_s . The number of all gaps in the segment $[1, 2, \dots, v]$ does not exceed $[v/2]$. Some conclusions are drawn from this theorem, in which reference is made to P. Medolaghi [Ann. Mat. Pura Appl. (2) 25, 179-217 (1897)] and to Yu. E. Penzov [see the preceding review and Mat. Sbornik N.S. 26(68), 161-182 (1950); these Rev. 12, 52]. D. J. Struik (Cambridge, Mass.).

Katsurada, Yoshie. On the connection parameters in a non-holonomic space of line-elements. J. Fac. Sci. Hokkaido Univ. Ser. I. 11, 129-149 (1950).

In einer Linienelementmannigfaltigkeit werde die infinitesimale Verschiebung eines Punktes durch die nichtintegrierbaren Ausdrücke (1) $ds^* = A^*(x, dx)$ definiert. Die A^* sind von einander unabhängige und in den dx^* von erster Ordnung homogene analytische Funktionen ihrer Argumente. Wegen der Homogenitätseigenschaft können die Gleichungen (1) in der Form (2) $ds^* = \lambda_a^*(x, dx) dx^a = A^*$ dargestellt werden. Eine Linienelementmannigfaltigkeit (x, x') mit einer durch $ds^* = \lambda_a^*(x, x') dx^a$ bestimmten Struktur wird als anholonom bezeichnet. Durch Auflösung von (2) gemäß $dx^a = \lambda_a^*(x, ds) ds^*$ findet man die zu den λ_a^* reziproken kontravarianten Vektoren λ^a . Wichtig ist es nun, dass man mit Hilfe dieser Vektoren die partiellen Ableitungen einer Funktion $f(x, x') (s' = ds^*/ds)$ nach den s^* definieren kann. Dabei treten die beiden wichtigen Anholonomitätsobjekte ω_{ab}^* und Ω_{ab}^* auf. Wesentlich für die ganze Arbeit ist die Erklärung einer Transformationstheorie. Sie geschieht folgendermaßen. An Stelle der Objekte A^* mögen Objekte \bar{A}^* durch (3) $\bar{A}^* = C^*(x, \bar{A})$ eingeführt werden. Es sei $\partial C^*/\partial \bar{A}^i = C^j$, dann werden die anholonomen Koordinatentransformationen durch $ds^* = C^*(x, ds/dt) ds^i$ definiert. Durch Auflösen von (3) nach den \bar{A}^i werden die $\bar{A}^i = \bar{C}^i(x, A)$ eingeführt, aus denen sich durch Differentiation nach den s^* die Objekte $\bar{C}_a^i = \partial \bar{C}^i / \partial s^a$ ergeben. Die C^* und \bar{C}_a^i legen die Transformationstheorie genau so fest, wie bei einer $x^i = x^i(x)$ holonomen Transformation die Größen $A_i^i = \partial x^i / \partial x^i$ und $A_{i,j}^i = \partial x^i / \partial x^j$. Dementsprechend wird das Transformationsgesetz für Tensoren sowie dass der Anholonomitätsobjekte ω_{ab}^* und Ω_{ab}^* angegeben. Das invariante Differential eines kontravarianten Vektors wird durch

$$\delta v^* = dv^* + \Gamma_{ab}^* v^a ds^b + C_{ab}^* v^a \delta s^b$$

definiert. Ist $v^* = s^*$ so wird noch die Forderung

$$\delta s^* = ds^* + G_a^* ds^a$$

gestellt, die als Basisübertragung bezeichnet wird. Die Γ_{ab}^* und G_a^* genügen dann einem gewissen Transformationsgesetz. Es werden dann zulässige Übertragungsparameter angegeben die sich aus dem Anholonomitätsobjekt Ω_{ab}^* und den G_a^* herleiten lassen. Aus dem Übertragungsparameter und den Anholonomitätsobjekten ergeben sich die Tor-

sions- und Krümmungstensoren. Es werden weiter durch $ds^* = \lambda_a^* dx^a$ bestimmte anholonome Untermannigfaltigkeiten X_n^* betrachtet und solche Transformationen der anholonomen Gesamtmannigfaltigkeit untersucht gegenüber denen, die X_n^* invariant sind. Auch die Krümmungs- und Torsionstheorie solcher Untermannigfaltigkeiten wird untersucht.

Ist ein Finslerscher Raum gegeben und ist $g_{ab}(x, x')$ der aus seiner Grundfunktion hergeleitete metrische Tensor und genügen die Funktionen $\lambda_a^* = \partial A^*(x, dx) / \partial (dx^a)$ den Relationen $g^{ab} \lambda_a^* \lambda_b^* = \delta^{ab}$ dann bezeichnet Verf. die Linienelementmannigfaltigkeit als euklidisch anholonom bezüglich den s^* . Es wird der Zusammenhang zwischen den anholonomen und Cartanschen Übertragungsparameter untersucht. Schliesslich werden in diesen Mannigfaltigkeiten euklidisch anholonome und insbesondere geodetisch anholonome euklidische Untermannigfaltigkeiten betrachtet.

O. Varga (Debrecen).

Katsurada, Yoshie. Non-holonomic system in a space of higher order. I. On the operations of extensors. J. Fac. Sci. Hokkaido Univ. Ser. I. 11, 190-217 (1951).

In Fortsetzung der obigen Arbeit werden anholonome Koordinatensysteme in Räumen höherer Ordnung, wie sie insbesondere von A. Kawaguchi betrachtet wurden, eingeführt. Ist ein Linienelementraum der Ordnung M gegeben, so ist er durch einen Punkt und Richtungen bestimmt, die durch die erweiterten Punkttransformationen bis zur M ten Ordnung festgelegt sind. Durch die erweiterten Punkttransformationen sind nach H. V. Craig Extensoren bestimmt. An Stelle der zueinander reziproken Vektorfelder λ_a^* und λ^a die im obigen Falle die anholonome Mannigfaltigkeit bestimmen, wird jetzt das anholonome System durch exkovariante Extensoren festgelegt. Grundsätzlich ist jetzt die Festlegung einer Transformationstheorie die sich aus derjenigen der erwähnten Extensoren ergibt. Nachdem so der Begriff der verallgemeinerten anholonomen Systeme festgelegt ist, werden eine Reihe von Kawaguchi betrachteten Operationen für Extensoren auf den anholonomen Fall übertragen. Wegen Einzelheiten muss wegen des nicht ganz einfachen Formelapparates, der hier nicht reproduziert werden kann, auf den Originaltext verwiesen werden.

O. Varga (Debrecen).

Katsurada, Yoshie. On the theory of non-holonomic systems in the Finsler space. Tôhoku Math. J. (2) 3, 140-148 (1951).

In einer n -dimensionalen Mannigfaltigkeit ist eine infinitesimale Verschiebung durch (1) $ds^* = A^*(x, dx)$ bestimmt. Die Funktionen $A^*(x, dx)$ sind in den dx^* von erster Dimension homogen, so dass die Formen auch durch

$$ds^* = \lambda_a^*(x, dx) dx^a; \quad \lambda_a^*(x, dx) = \partial A^* / \partial (dx^a)$$

dargestellt werden können. In der Linienelementmannigfaltigkeit (x, x') sei ein spezielles anholonomes System durch (2) $ds^* = \lambda_a^*(x, x') dx^a$ definiert. Die durch (2) bestimmte Verschiebung fällt dann und nur dann mit der durch (1) bestimmten zusammen, falls in der Richtung des Linienelementes verschoben wird. Führt man die zu den λ_a^* reziproken kontravarianten Vektoren λ^a ein, so können die Anholonomitätsobjekte, auf die in anholonomen Bezugssysteme übliche Art durch Bildung der zweiten Ableitungen und Vertauschung derselben gewonnen werden.

Eine Transformationstheorie ergibt sich falls man im (1) statt A^* neue Objekte \bar{A}^* einführt. Verf. leitet in erster Reihe das Transformationsgesetz der Anholonomitäts-

objekte her. Weiter wird der Formelapparat der kovarianten Ableitung für anholonome Bezugssysteme entwickelt. Für den Fall in dem die Linienelementmannigfaltigkeit ein Finslerscher Raum ist, wird das invariante Cartansche Differenzial eines Vektors für anholonome Bezugssysteme erweitert. Zum Schluss wird die Krümmungs- und Torsionstheorie behandelt, wobei Verf. ausser Finslerschen Räumen auch allgemeine Linienelementmannigfaltigkeiten betrachtet. *O. Varga (Debrecen).*

Vasil'ev, A. M. On algebraic operations applicable in differential geometry. Doklady Akad. Nauk SSSR (N.S.) 82, 509-511 (1952). (Russian)

An outline of the algebraic operations involved in the study of differential-geometric objects. The study of representations of the analytic group P_n of an n -dimensional space is equivalent to that of finite groups $D_{(n,\nu)}$ of extensions up to the order ν . Thus the study of differential objects is reduced to extensions and the classical theory of rational integral invariants. *M. S. Knebelman.*

Lichnerowicz, André. Sur les variétés symplectiques. C. R. Acad. Sci. Paris 233, 723-725 (1951).

An almost Kähler manifold V_{2n} (definable by n complex valued 1-forms θ^a) is locally Kähler if the associated 2-form has covariant derivative 0; a necessary and sufficient condition for this is complete integrability of the θ^a . A form is proper if all its terms have the same parity in the θ^a . In addition to the usual operators * , d , δ , L , Δ an operator I ($I^2 = (-1)^p$) is defined (using F) which sends (proper) p -forms into (proper) p -forms. Theorem: φ and $I\varphi$ are harmonic if φ can be generated from proper closed forms of class 0 by using L (and addition). This has applications for real cohomology, giving lower bounds for the Betti numbers in terms of closed proper forms of class 0. Another application gives an existence theorem: If on a compact almost Kähler manifold there exists a form with vanishing covariant derivative (different from the volume element), then there exist closed proper forms of class 0. *H. Samelson.*

Libermann, Paulette. Sur les variétés presque complexes V_{2n} munies d'un champ de n -éléments réels. C. R. Acad. Sci. Paris 233, 1571-1573 (1951).

The author considers in a manifold V_{2n} , an almost complex structure and a field of real n -planes. Describing this by means of complex 1-forms, one obtains a system of structure equations, an affine connection, a curvature and a torsion. Theorem I: The given real field and the associated imaginary field are of the form $dy_1 = \dots = dy_n = 0$, resp. $dx_1 = \dots = dx_n = 0$, for suitable complex coordinates z_1, \dots, z_n if curvature and torsion vanish. Similar theorems concern the complete integrability of either the real or the imaginary field, and the effects of varying the fields by multiplication by a factor $\exp(i\varphi)$. *H. Samelson.*

Libermann, Paulette. Formes différentielles sur une variété symplectique. C. R. Acad. Sci. Paris 234, 395-397 (1952).

In $2n$ -space R^{2n} an exterior 2-form Ω and a quadratic form F are given (by $\sum \omega_j \wedge \omega_{j+n}$ and $\sum (\omega_j)^2$, where the ω 's are $2n$ independent linear forms); this defines a complex structure on R^{2n} . The author introduces the notion of adjoint of a form with respect to Ω , using notions of Grassmann algebra, and considers this concept together with the usual operators of the theory of forms on complex manifolds. The operator Δ turns out to depend on Ω only, not on F . Effective

forms and class of a form are introduced, and the decomposition theorem is stated [Eckmann and Guggenheimer, same C. R. 229, 489-491 (1949); these Rev. 11, 212]; the connection with the algebraic theorems of Lepage [Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 325-345 (1949); these Rev. 11, 308] is pointed out. The theory is applied to forms on a manifold. There one has now two co-differentials δ and $\bar{\delta}$, corresponding to F and Ω . For the symplectic case ($d\Omega=0$) several theorems are stated concerning effective forms, and forms harmonic with respect to Ω .

H. Samelson (Ann Arbor, Mich.).

Libermann, Paulette. Sur les structures presque quaternioniennes de deuxième espèce. C. R. Acad. Sci. Paris 234, 1030-1032 (1952).

The author considers three automorphisms I, J, K , of Euclidean R^{2n} , satisfying $I^2 = -1, J^2 = 1, IJ = -JI, K = IJ$; this is equivalent to specifying three mutually supplementary n -planes. For even $n=2p$ this makes R^{2n} into a module M^p over the quaternions of second kind, and determines the linear quaternionic and the linear unitary quaternionic group of second kind, the latter being isomorphic with the symplectic real group. In a manifold V_{2n} one now assumes given a field I, J, K of such operators, or equivalently a triple of fields of n -planes, mutually supplementary at each point. This is an infinitesimal regular structure in the sense of Ehresmann [see the following review]. For even n one gets an almost quaternionic structure of second kind. It is possible to describe this by complex 1-forms, and to derive structure equations. Integrable structures are locally affine, as for (ordinary) quaternionic structures. *H. Samelson.*

Ehresmann, Charles. Structures locales et structures infinitésimales. C. R. Acad. Sci. Paris 234, 587-589 (1952).

This continues three earlier notes [same C. R. 233, 598-600, 777-779, 1081-1083 (1951); these Rev. 13, 386, 584]. The author sketches a general theory of local and infinitesimal structures. Local structures are defined roughly as structures (in the sense of Bourbaki) on a set E , which induce the same kind of structure on the open sets of a topology on E and which define a structure on a set M if structures are given on the sets M_i of a covering of M , provided the given structures are compatible, i.e., agree on the intersections $M_i \cap M_j$. Many associated concepts, e.g., local automorphisms are discussed. Infinitesimal structures pertain to differentiable manifolds and involve cross sections in prolongements. This is related to Golab's "pure differential object" [Ann. Soc. Polon. Math. 19, 7-35 (1947); these Rev. 9, 206]. Special cases and related concepts are considered. *H. Samelson (Ann Arbor, Mich.).*

Ehresmann, Charles. Les prolongements d'une variété différentiable. IV. Éléments de contact et éléments d'enveloppe. C. R. Acad. Sci. Paris 234, 1028-1030 (1952).

The author continues his study of prolongements of a differentiable manifold [see the preceding review and references cited there]. For a fiber bundle $E(B, F, G, H)$ the associated groupoid Π consists of all the admissible maps between fibers. Π has a fiber bundle structure with $B \times B$ as base, G as fiber, $G \times G$ as group. Orbits (classes of intransitivity) and invariant sets (unions of orbits) under Π are introduced. If φ is a map of G into the group \hat{G} of automorphisms of a space \hat{F} , one has an induced bundle $\hat{E}(B, \hat{F}, \hat{G}, \hat{H})$, and an induced map $\hat{\varphi}$ of Π into $\hat{\Pi}$. A map ψ :

$E \rightarrow \bar{E}$ is called covariant if $\psi \cdot \theta = \psi(\theta) \cdot \psi$ for every $\theta \in \Pi$. These concepts are applied to prolongements of manifolds. General definitions of contact element, envelope element and others are given.

H. Samelson (Ann Arbor, Mich.).

Guggenheimer, H. Über komplex-analytische Mannigfaltigkeiten mit Kählerscher Metrik. Comment. Math. Helv. 25, 257-297 (1951).

This is a detailed account of results announced earlier by Eckmann and the author [C. R. Acad. Sci. Paris 229, 464-466, 489-491, 503-505, 577-579 (1949); these Rev. 11, 212]. It is mainly concerned with generalizations to Kähler manifolds of theorems of Hodge on the homology properties of complex algebraic varieties without singularities. That such a generalization is possible was indicated by A. Weil [Comment. Math. Helv. 20, 110-116 (1947); these Rev. 9, 65].

As in Hodge's work the main tool is the theory of harmonic integrals, whose formalism is developed here in a slightly different fashion to suit the Hermitian nature of the metric. We recall that a Kähler manifold is a complex Hermitian manifold of (complex) dimension m , such that, if $\sum_{i,j=1}^m g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ defines the Hermitian metric in the local coordinates z_i , the corresponding exterior differential form $\Omega = \sum_{i,j=1}^m g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ is closed. On the manifold the following operators on exterior differential forms can be defined: 1) the star operator $*$; 2) the operator C , depending on the complex structure; 3) the operators L, Δ , of which the first is essentially the multiplication of a differential form by Ω , while the second is defined in terms of L and $*$. Relations between these operators are established. In particular, it is proved that they all carry harmonic forms into harmonic forms. Among the topological consequences is, for instance, the theorem (generalizing one of Lefschetz) that, if R_p denotes the p th Betti number of a compact Kähler manifold, R_p is even if p is odd, and $R_p \geq R_{p-2}$ if $p \leq m$. Among the theorems depending on the Kählerian nature of the metric is a generalization of Hodge's decomposition theorem:

$$H^r = H_0^r + L H_0^{r-2} + \dots + L^q H_0^{r-2q}, \quad r \leq m, \quad q = [\frac{1}{2}r],$$

where H^r is the group of all harmonic r -forms and H_0^r the group of all effective harmonic r -forms (that is, harmonic

r -forms φ satisfying $\Delta\varphi=0$). In an appendix it is proved that results such as the decomposition theorem, which do not depend on the complex structure, can be generalized to an even-dimensional Riemann manifold with an exterior differential form of zero covariant derivative and everywhere of rank $2k$.

S. Chern (Chicago, Ill.).

Yano, Kentaro, and Ohgane, Masayoshi. On unified field theories. Ann. of Math. (2) 55, 318-327 (1952).

If a vectorfield $v^k, k=0, 1, \dots, n$ is given in a V_{n+1} it is possible to define in each point n other vectors perpendicular to v^k . In this way we get a V_{n+1}^* and an anholonomic coordinate system adapted to this V_{n+1}^* . The components of the anholonomy object $\Omega_{\alpha\beta}^{\gamma}, \alpha, \beta=0, 1, \dots, n$ have very simple values. The only non-vanishing ones are $\varphi_{ji} = \Omega_{ji}^0$ and $\tau_j = -\Omega_{ji}^0$. In the V_{n+1}^* a connexion is induced and we have two curvature tensors (affinors) $K_{\beta\gamma\delta}^{\alpha}$ and $R_{\beta\gamma\delta}^{\alpha}$ (indices in Princeton-setting). A lot of interesting relations can be derived. Besides V_{n+1} and V_{n+1}^* we can construct a V_n by reduction (Zusammenlegung) of the V_{n+1} with respect to the congruence of the streamlines of v^k . Only those quantities of V_{n+1} that are G -invariant, that is, whose Lie derivative with respect to v^k vanishes, are quantities of this V_n . The tensor $G_{\alpha\beta}$ induces a fundamental tensor in V_n if and only if the V_{n+1} -part of $G_{\alpha\beta}$ is G -invariant. This gives a set of conditions in which τ_j plays an important role. The authors take 5 pages for all these definitions and calculations and they succeed in giving them in a clear and satisfactory way. From now on physics begins. It is shown that the proposed differential geometry can be interpreted so as to contain all the geometries appearing in the five-dimensional unified field theories in the past. But the authors claim that the theory gives more because it is the only one in which a vector k_i plays a role. This vector is essentially the same as τ_i and passes into τ_i if v^k happens to have unit length. Moreover, they say that the theory suggests a natural generalization of the six-dimensional unified field theorems proposed by Hoffman, Podolanski and one of the authors themselves, but the proof of this assertion seems to be postponed to a future publication.

J. A. Schouten (Epe).

NUMERICAL AND GRAPHICAL METHODS

Powell, E. O. A table of the generalized Riemann zeta function in a particular case. Quart. J. Mech. Appl. Math. 5, 116-123 (1952).

The Hurwitz zeta function, defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$$

when $\Re(s) > 1$ and $\Re(a) > 0$, occurs in various connections; for $\Re(a) > 0$, the function, aside from a simple pole at $s=1$, has an analytic continuation over the whole s -plane. It reduces to the Riemann zeta function when $a=1$, and $\zeta(2, a) = [\Gamma'(a)/\Gamma(a)]'$; furthermore, it occurs in number theory because of its relation to the Dirichlet L -function: $\sum_{n=1}^{\infty} \chi(n) n^{-s} = k^{-s} \sum_{r=1}^k \chi(r) \zeta(s, r/k)$ where χ is a character modulo k . The author's interest in the function $\zeta(\frac{1}{2}, a)$ stems from the fact that it is a solution of the difference equation $f(a+1) - f(a) + a^{1/2} = 0$. The author provides a 10-decimal place table of $\zeta(\frac{1}{2}, a)$ for $a=1.00(.01)2.00(.02)5.00(.05)10.00$ together with modified second central differences; he also makes reference to unpublished 5-decimal tables of $\zeta(s, a)$

for $s=-10.0(.1)0, a=.0(.1)2.0$ and of $(s-1)\zeta(s, a)$ for $s=.0(.1)1.0, a=.0(.1)2.0$. L. Schoenfeld (Urbana, Ill.).

David, F. N., and Kendall, M. G. Tables of symmetric functions. II, III. Biometrika 38, 435-462 (1951).

Part I of this paper [Biometrika 36, 431-449 (1949); these Rev. 11, 488] contained tables giving the expressions for the monomial symmetric functions $\sum x_1^{a_1} x_2^{a_2} \dots$ in terms of the sums $s_r = (r) = \sum x_i^r$ of r th powers and inverse tables. The present tables are of two sorts. Those of the first sort give expressions for the monomial symmetric functions in terms of the elementary or unitary symmetric functions $a_r = (1^r) = \sum x_1 x_2 \dots x_r$ and inversely. The tables of the second sort give expressions for the monomial symmetric functions in terms of the "homogeneous product-sums" h_r defined in terms of the a 's by the identity

$$(1 + h_1 t + h_2 t^2 + \dots)(1 - a_1 t + a_2 t^2 - \dots) = 1$$

and inversely. All tables are for symmetric functions of weights ≤ 12 . The latter tables appear to be new.

D. H. Lehmer (Los Angeles, Calif.).

Ozorio de Almeida, Miguel. Sur le calcul des paramètres des équations de la forme $y = a + bx + cx^2 + \dots$ destinées à représenter les valeurs des variables données par des observations ou des expériences. I. Cas où les valeurs de la variable indépendante x sont en progression arithmétique. Anais Acad. Brasil. Ci. 23, 421-428 (1951).

Ozorio de Almeida, Miguel. Sur le calcul des paramètres des équations de la forme $y = a + bx + cx^2 + \dots$ destinées à représenter les valeurs des variables données par des observations ou des expériences. II. Cas où les valeurs de la variable indépendante se suivent dans un ordre arbitraire. Anais Acad. Brasil. Ci. 23, 429-436 (1951).

Cintra, Horácio. Sur le calcul des paramètres des équations de la forme $y = a + bx + cx^2 + \dots$ destinées à représenter les valeurs des variables données par des observations ou des expériences. III. Essai d'une théorie d'ajustement paramétrique. Anais Acad. Brasil. Ci. 23, 437-441 (1951).

Michel, J. G. L. Direct calculation of smooth gunnery range tables. Quart. J. Mech. Appl. Math. 5, 124-128 (1952).

As opposed to the slow and awkward practice of inverse two-variable interpolation, the author proposes as a device for inverting any pair of non-linear functions of two variables, to use the linearity of the relationships among the differentials, and by integration of the inverse of the Jacobian to obtain the desired inverse pair. He illustrates the value of this method for a typical problem of ballistics.

A. A. Bennett (Providence, R. I.).

Casale, Francesco. Calcolo approssimato delle radici reali di una equazione. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 727-734 (1950).

Let $y = f(x)$ be a function continuous with derivatives of all orders and $f'(x) \neq 0$ in the interval x_1 to x_2 . Let $f(x_1) = a$, $f(x_2) = b$, where a and b have opposite signs. By the formula

$$x = x_1 - a[1/f']_{x_1} + a^2/2![-f''/f'^2]_{x_1} - a^3/3![(3f''' - f''f'')/f'^3]_{x_1} + a^4/4![(10f''f''' - f'^2f^{(4)} - 15f''^2)/f'^4]_{x_1} - \dots$$

the roots of $f(x) = 0$ may be rapidly computed and also the error involved. Examples are given which illustrate the method.

E. Frank (Chicago, Ill.).

Kaplan, E. L. Numerical integration near a singularity. J. Math. Physics 31, 1-28 (1952).

Au voisinage d'une singularité de forme connue, on peut utiliser des formules de quadrature approchée basées sur l'interpolation non pas au moyen de polynômes mais au moyen de fonctions ayant des singularités de forme convenable. L'auteur donne des tables de tels coefficients pour les intégrales

$$\int_{r_1}^{r_2} f(x) dx, \quad f(x) = x^n A(x), \quad A(x) \text{ régulier, } n = \pm \frac{1}{2};$$

les valeurs intervenant dans ces formules sont soit celles de $f(x)$ soit celles de $A(x)$; $r_2 = 2 + r_1$ (dans quelques cas $r_2 = 1 + r_1$); $r_1 = 0(1)18, 14$ déc.; la formule utilise 3 ou 5 points (équidistance 1). Les tables sont également données pour une fonction $A(x^2)$. Tables analogues pour des intégrales de la forme

$$\int_{r_1}^{r_2} x^n [A(x) \log x + B(x)] dx, \quad A(x), B(x) \text{ réguliers, } n = 0, 1;$$

$r_2 = r_1 + 1$; $r_1 = 0(1)20$; 10 déc.; la formule utilise 6 points (équidistance 1).

J. Kuntzmann (Grenoble).

Lotkin, Mark. A new integrating procedure of high accuracy. J. Math. Physics 31, 29-34 (1952).

L'auteur donne une méthode d'intégration approchée des équations différentielles du premier ordre basée sur la formule d'Euler-MacLaurin. Cela conduit pour chaque pas aux itérations:

$$y_{i,j+1} = y_{i,j} + \frac{1}{2}x(f_{i,j} + f_{i,j+1}) + \frac{1}{10}x^2(f'_{i,j} - f'_{i,j+1}) + \frac{1}{120}x^3(f''_{i,j} + f''_{i,j+1}).$$

Lorsque les itérations ont été poussées assez loin l'erreur est du 7ème ordre (erreur de méthode due au reste de la formule d'Euler-MacLaurin). La méthode s'applique naturellement aux systèmes mis sous forme canonique. Un exemple est donné relatif à l'équation $y' = x + y$.

J. Kuntzmann.

Young, David. An error bound for the numerical quadrature of analytic functions. J. Math. Physics 31, 42-44 (1952).

Dans une note précédente [G. Birkhoff et D. Young, même J. 29, 217-221 (1950); ces Rev. 12, 445] l'auteur a donné une formule de quadrature approchée à cinq points pour une fonction analytique utilisant des résultats de Montel [Ann. Scuola Norm. Super. Pisa (2) 1, 371-384 (1932); J. Math. Pures Appl. (9) 16, 219-231 (1937)]; il donne ici une borne supérieure de l'erreur. Le résultat de Montel peut d'ailleurs être étendu à l'interpolation à points répétés.

J. Kuntzmann (Grenoble).

Flügge-Lotz, I. Mathematical improvement of method for computing Poisson integrals involved in determination of velocity distribution on airfoils. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2451, 84 pp. (1951).

The problem is to evaluate the principal value of improper integrals of the form $\int_a^b (x - x_0)^{-1} \sigma(x) dx$ where $a < x_0 < b$ and $\sigma(x)$ is smooth at x_0 . Previous methods are discussed which use equal intervals and lead to formulas involving fixed linear combinations of the values σ_n of $\sigma(x)$ at the division points, the coefficients of which can be tabulated once for all for a given number of divisions. The present paper uses intervals of variable size, smaller near x_0 , and leads to linear combinations of σ_n with coefficients which need not themselves be fixed but which can always be taken from a table of the fixed function $j(y, z) = \int_0^1 t^{-1} x^{-1} dx$, where again the integral is a principal value. Four place tables of j as a function of $w = y/(z - y)$ are given for $\pm w = 190(1.0)90(5)40(1)20(01)1(001)0$. Values of $j^* = 1 - wj$ are also given for a similar range.

P. W. Ketchum (Urbana, Ill.).

Berry, V. J., and de Prima, C. R. An iterative method for the solution of eigenvalue problems. J. Appl. Phys. 23, 195-198 (1952).

Für das Eigenwertproblem

$$D(\lambda) = \int_0^1 (pu^2 + gu^3) dx \rightarrow \text{Min},$$

$$u(0) = 0, \quad \int_0^1 wu^2 dx = 1, \quad p > 0, \quad w > 0 \quad \text{für } 0 < x \leq 1$$

wird zur Auffindung des Eigenwertes ein Iterationsverfahren angegeben. Man denke sich für die Lösung der zum Eigen-

wertproblem zugehörigen Differentialgleichung unter Berücksichtigung der beiden Nebenbedingungen den Wert von $D(\lambda)$ berechnet (ohne Berücksichtigung der freien Randbedingung am Ende). Zunächst wird bewiesen, dass $D(\lambda)$ eine monoton wachsende Funktion von λ ist. Denkt man sich $D(\lambda)$ in einer (λ, D) -Ebene als Kurve aufgetragen, dann entsprechen den Eigenwerten die stationären Werte von $D(\lambda)$, also den Schnittpunkten mit der Geraden $\lambda=D$. Bezeichnet man nun mit

$$G(\lambda) = p(1)u(1, \lambda)u'(1, \lambda)$$

so ist das Iterationsverfahren gekennzeichnet durch

$$\lambda_n^{(k+1)} = \lambda_n^{(k)} + G(\lambda_n^{(k)}).$$

Unter Berücksichtigung der Monotonie von $D(\lambda)$ ergibt sich die Konvergenz des Verfahrens, wenn der Ausgangspunkt zwischen zwei zunächst gelegenen Wendepunkten angenommen wird. Das Verfahren ist insbesondere geeignet für die Ermittlung von höheren Eigenwerten. Die Methode wird numerisch am Beispiel der Besselfunktion erster Ordnung vorgeführt, und zwar ergibt sich für $\lambda_4^{(4)}$ eine Übereinstimmung in 6 Stellen.
P. Funk (Wien).

Muhin, I. S. Application of the Markov-Hermite interpolation polynomials for numerical integration of ordinary differential equations. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 16, 231-238 (1952). (Russian)

The author employs the Markov-Hermite polynomial interpolation formula to derive several pairs of formulas useful in the numerical integration of differential equations, especially if solved by methods proposed by the reviewer. Of each pair the first is an open type, or extrapolatory, integration formula, the second a closed type, or interpolatory, integration formula. The paper contains pairs with first derivatives only, pairs with higher order derivatives (one being a special case of Obrechhoff's formula) and pairs suited to second order equations lacking first derivatives. Most of the formulas are well known. Rigorous error terms are given for each.
W. E. Milne (Corvallis, Ore.).

Leutert, Werner. On the convergence of unstable approximate solutions of the heat equation to the exact solution. *J. Math. Physics* 30, 245-251 (1952).

The solution $u(x, t)$ of the problem defined by the differential equation $\partial u / \partial t = \partial^2 u / \partial x^2$, the boundary conditions $u(0, t) = u(1, t) = 0$, $t \geq 0$, and the initial condition $u(x, 0) = f(x)$, $0 < x < 1$, is compared with the solution of an analogous problem for the finite difference equation

$$v(x, t + \Delta t) - v(x, t) = r[v(x + \Delta x, t) + v(x - \Delta x, t) - 2v(x, t)]$$

where $(\Delta x)^2 r = \Delta t$. It is known that for $r > \frac{1}{2}$ the solution of such a finite difference problem is unstable, in the sense in which this term is commonly used in numerical analysis. The author proves that, in spite of this fact, there exist, for every fixed r , solutions $v(x, t, \Delta x)$ of the difference equation that converge to the solution $u(x, t)$ of the differential problem, as $\Delta x \rightarrow 0$. The functions $v(x, t, \Delta x)$ satisfy strictly the boundary conditions $v(0, t, \Delta x) = v(1, t, \Delta x) = 0$, $t > 0$, but the initial condition at $t = 0$ can in general be satisfied only in the limit as $\Delta x \rightarrow 0$. The function $f(x)$ is assumed to be sectionally continuous and to have one-sided first derivatives. Thanks to this assumption $f(x)$, and hence the solution of the differential problem, can be represented by a convergent Fourier series. The functions $v(x, t, \Delta x)$ are then defined as certain related finite trigonometric polynomials.
W. Wasow (Los Angeles, Calif.).

Leutert, Werner, and O'Brien, George G. On the convergence of approximate solutions of the wave equation to the exact solution. *J. Math. Physics* 30, 252-256 (1952).

A result analogous to that of the paper reviewed above is derived concerning the relation between the solution $u(x, t)$ of the hyperbolic differential equation $\partial^2 u / \partial t^2 = \partial^2 u / \partial x^2$ satisfying the boundary conditions $u(0, t) = u(1, t) = 0$, $t \geq 0$, and the initial conditions $u(x, 0) = f(x)$, $[\partial u(x, t) / \partial t]_{t=0} = 0$, $0 \leq x \leq 1$, and certain solutions of the finite difference equation

$$v(x, t + \Delta t) + v(x, t - \Delta t) - 2v(x, t) = r[v(x + \Delta x, t) + v(x - \Delta x, t) - 2v(x, t)],$$

where $\Delta x \cdot r = \Delta t$. These latter solutions are constructed in such a way that they satisfy strictly the boundary conditions and converge for any fixed r to $u(x, t)$, as $\Delta x \rightarrow 0$. The initial conditions, however, can, for $r > 1$, be satisfied only in the limit, as $\Delta x \rightarrow 0$.
W. Wasow (Los Angeles, Calif.).

Cesarina, Tibiletti. Sull'integrazione grafica delle equazioni differenziali. *Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat.* (3) 13(82), 451-472 (1949).

This paper gives a method of graphical integration for differential equations of first and second order. For the first order equation $y' = f(x, y)$ the method consists in using certain curves (logarithmic spirals, hyperbolas, cycloids, general conics), for which there is a simple construction, to approximate an integral curve through a point A_0 in such a way that the coordinates and differential elements up through a certain order r (with $r = 3$ or 4) agree at A_0 . An analogous process is indicated for the differential equation of second order. Next the author considers the error, or what amounts to the same thing, how far from the point A_0 one can safely go along one of these approximating arcs. Finally he considers certain processes of averaging to improve these jointed approximating arcs.
W. E. Milne.

Pipes, Louis A. The reversion method for solving non-linear differential equations. *J. Appl. Phys.* 23, 202-207 (1952).

The author explains and illustrates a method for the approximate solution of a class of non-linear differential equations treated originally by C. E. Van Orstrand [*Philos. Mag.* (6) 19, 366-376 (1910)]. The equations are of the form $a_1 y + a_2 y^2 + \dots + a_n y^n = k \varphi(t)$ where t is the independent variable, k is a constant, and the a 's are functions of the operator $D = d/dt$. The process consists in determining one by one the coefficients A_i in an assumed solution of the form $y = A_1 k + A_2 k^2 + \dots$. The methods of the Laplace transform theory are used to solve the linear differential equations which determine the A 's. The method is illustrated by seven worked examples drawn mainly from circuit theory.
W. E. Milne (Corvallis, Ore.).

Salpeter, E. E. Wave functions in momentum space. *Physical Rev.* (2) 84, 1226-1231 (1951).

Techniques are described for the numerical application of an iteration method to the solution of the integral equation satisfied by the radial part of the wave function in momentum space. Essential features of the method are the choice of suitable initial trial functions, and the use of properly chosen values of the argument to provide tolerably accurate numerical integrations and useful information about the wave function. It is indicated that this sort of

method can be used to determine phase shifts in scattering problems, as well as energy values for stationary states. The possibility of applications to cases in which there is a "tensor force" in addition to the central force is indicated.

W. H. Furry (Cambridge, Mass.).

Capra, Vincenzo. Sull'integrazione delle equazioni differenziali della balistica mediante nomogrammi. Univ. e Politecnico Torino. Rend. Sem. Mat. 10, 235-241 (1 plate) (1951).

The author assumes ballistic equations with exponential air density (ignoring temperature effects upon the Mach number). As independent variable he adopts $\log \tan (\pi/4 + \theta/2)$, (the familiar antigudermannian of the angle of inclination). He constructs a nomogram with 5 vertical and 7 horizontal scales, by means of which one may obtain successive tangents enveloping the trajectory. Four curves are to be thus plotted. Straight edges must be used not only to join finitely accessible points but also to draw vertical and horizontal lines as usual in analytic geometry. The author prefers graphical to numerical methods.

A. A. Bennett.

Aparo, Enzo, e Dainelli, Dino. L'EDSAC, una moderna macchina calcolatrice elettronica. Ricerca Sci. 22, 186-201 (1952).

van der Poel, W. L. A simple electronic digital computer. Appl. Sci. Research B. 2, 367-400 (1952).

Hermes, Hans. Maschinen zur Entscheidung von mathematischen Problemen. Math.-Phys. Semesterber. 2, 179-189 (1952).
Expository paper.

Štykan, A. B. An integrating mechanism of Leibniz. Uspehi Matem. Nauk (N.S.) 7, no. 1(47), 191-194 (1952).
(Russian)

Knappe, W. Eine neue Zwangsführung zum Nyströmschen Stieltjesplanimeter. Z. Angew. Math. Mech. 32, 84-85 (1952).

Breitling, Paul. Zirkel als Planimeter. Schweiz. Z. Vermessg. Kulturtech. 50, 94-98 (1952).

Sprague, R. E. Fundamental concepts of the digital differential analyzer method of computation. Math. Tables and Other Aids to Computation 6, 41-49 (1952).

Korn, Granino A. The difference analyzer: a simple differential equation solver. Math. Tables and Other Aids to Computation 6, 1-8 (1952).

Brooks, F. E., Jr., and Smith, H. W. A computer for correlation functions. Rev. Sci. Instruments 23, 121-126 (1952).

Laville, Gaston. Méthode graphique applicable à l'analyse harmonique et au calcul symbolique. C. R. Acad. Sci. Paris 234, 1728-1730 (1952).

Broomall, John, and Riebmán, Leon. A sampling analogue computer. Proc. I. R. E. 40, 568-572 (1952).

Perdok, W. G. Analysis of crystal structures with I.B.M. (Hollerith) punched card machines. Nederl. Tijdschr. Natuurkunde 18, 49-68 (1952). (Dutch)

Boni, Alessandro. Studi sul calcolo meccanico compiuti presso l'Istituto Nazionale per le Applicazioni del Calcolo. Ricerca Sci. 22, 429-433 (1952).

Batschelet, Eduard, and Striebel, Hans Rudolf. Nomogramm zur Bestimmung der reellen und komplexen Wurzeln einer Gleichung vierten Grades. Z. Angew. Math. Physik 3, 156-159 (1952).

ASTRONOMY

Jeffreys, Harold. On the figure of a planet with homogeneous shell and core. Monthly Not. Roy. Astr. Soc. 111, 410-412 (1951).

Using a first order theory the author solves without the Radau approximation the equations of equilibrium of a two-layer rotating body with uniform densities in the shell and core and then compares his results with those of the approximation method. The two methods show appreciable variance when the central condensation is stronger than in the Earth.

R. G. Langebartel (Urbana, Ill.).

de Jekhowsky, Benjamin. Détermination des orbites paraboliques à partir de plusieurs observations. C. R. Acad. Sci. Paris 233, 779-781 (1951).

Olbers' method of determining a parabolic orbit employs the relation $p_1 = M p_1$ where

$$M = -c_1[\bar{p}_1^* \cdot \bar{p}_2^* \times \bar{U}] / c_2[\bar{p}_3^* \cdot \bar{p}_4^* \times \bar{U}]$$

and \bar{U} must satisfy certain conditions. The author generalizes this relation so that, if there are more than three observations, the geocentric distance at any other time may be expressed in the form $p_i = M p_{i-2} + N_i$, where M_i and N_i have elaborate formulas involving numerous vector cross products and summations. The triangle ratios c_i are replaced by the first two terms of their series expansions. Considering

all the computations of auxiliary quantities which are required, it is difficult to see where this device could profitably be used. The basic concept is considerably obscured by the complicated notation employed.

P. Herget.

de Jekhowsky, Benjamin. Sur la détermination des distances géocentrique p et héliocentrique r d'astéroïdes dans la méthode de Laplace. C. R. Acad. Sci. Paris 234, 1436-1438 (1952).

Rabe, W. Neue Methoden zur Bahnbestimmung und Bahnverbesserung visueller Doppelsterne. Astr. Nachr. 280, 1-23 (1951).

Of visual double stars of long period the only accurate data obtainable from the observations are the apparent relative coordinates and their first derivatives, for a single epoch. Three additional data are required for a complete solution. In some instances these may be obtained from the higher derivatives, but the practical value of this approach is limited.

The methods developed in this paper avoid the use of the higher derivatives. Instead, a trial orbit is introduced. This orbit satisfies the four accurately available data while the three remaining data are supplied more or less arbitrarily, guided by a judicious use of all the observational data. These

may be the eccentricity, the period of revolution and the time of periastron passage, but various other choices may be made. After a comparison of the observations with the trial orbit a differential correction may follow. The amount of computing required is small compared with other methods and is reduced by the use of auxiliary tables furnished in the article.
D. Brouwer (New Haven, Conn.).

Luchak, George. A fundamental theory of the magnetism of massive rotating bodies. *Canadian J. Physics* 29, 470-479 (1951).

This is an attempt to explain Blackett's relation between the magnetic moment and angular momentum of the earth, sun and five stars by a generalization of phenomenological electromagnetic theory to include gravitational effects. The theory is distinguished from other such attempts by the introduction of what may be called a dibaric constant ξ which, like the dielectric constant, may vary with physical conditions. The theory is based on Nordström's five-dimensional flat-space unification of gravitation and electromagnetism (which does not yield the three famous criteria for a gravitational theory). Blackett's formula certainly emerges if $\xi \sim 10$ in the core of the earth and ~ 1.6 in the stars, though it is difficult to understand the physical significance of this quantity.
H. C. Corben (Genoa).

Öpik, E. J. Rotational currents. *Monthly Not. Roy. Astr. Soc.* 111, 278-288 (1951).

The author considers a star of small oblateness rotating with constant angular velocity, and assumes that (contrary to the assumption leading to von Zeipel's paradox [von Zeipel, *Festschrift für H. von Seeliger*, Springer, Berlin, 1924, pp. 144-152]) the energy generated per unit mass and time is not constant over a surface of constant potential. Then the average energy generation over such a surface is shown to be not affected by rotation. The balancing convective circulation is derived and it is concluded that the mixing efficiency of rotational currents as compared to the speed of nuclear reactions is practically negligible for stars rotating under moderate speeds.
R. G. Langebartel.

Schatzman, Evry. Sur la stabilité de certains modèles de planètes. *Acad. Roy. Belgique. Bull. Cl. Sci.* (5) 37, 599-609 (1951).

The stability of models of planets with the density a step-function of the pressure is considered, following Ramsey and the reviewer [cf. Lighthill, *Monthly Not. Roy. Astr. Soc.* 110, 339-342 (1950); these Rev. 12, 641].
M. J. Lighthill (Manchester).

Pecker, Jean-Claude. Contribution à la théorie du type spectral. I. Introduction générale. La construction des modèles d'atmosphères. *Ann. Astrophysique* 13, 294-318 (1950).

This summarizing investigation is devoted to a fairly detailed and elementary exposition of the problems of the construction of stellar model-atmospheres and of the formation of spectral lines, which are essential for a theoretical interpretation of stellar spectra. After a recapitulation of the fundamental concepts and definitions, the author sets up the well-known equations of radiative transfer and proceeds to solve them, for the case of a constant as well as variable absorption coefficients, by the methods initiated by Chandrasekhar [cf. his *Radiative transfer*, Oxford Univ. Press, 1950; these Rev. 13, 136] and Kourganoff. This discussion is followed by a summary of the formulae which can

be used to compute the net flux, the intensity, as well as the pressure of the emergent radiation. The tables constructed by Kourganoff and Strömgren for the computation of the flux and intensity are extended to include also the corresponding radiation pressure.
Z. Kopal (Manchester).

Pecker, Jean-Claude. Contribution à la théorie du type spectral. II. La polarisation par les électrons libres et le type spectral. *Ann. Astrophysique* 13, 319-336 (1950).

The author investigates the effects of polarization in a stellar atmosphere in which the total opacity is due to the atomic absorption (the coefficient of which is regarded to be independent of the wave-length) as well as to the scattering of light on free electrons. This latter scattering will produce polarization; and the author sets out to investigate the transfer of both the polarized and unpolarized components of the total radiation. The corresponding equations of transfer are solved numerically to a "first" and "second" approximation along the lines initiated by Chandrasekhar. An application of the results to Lyot's measurements of the polarization of solar radiation is shown to lead to a satisfactory agreement between theory and observation; and an even better agreement is claimed to exist between theory and the degree of polarization as measured by Janssen and Hiltner at the limb of the B-type components of eclipsing systems of U Sagittae (B9) and RY Persei (B4). (Similar results have been published almost simultaneously by A. D. Code [Astrophys. J. 112, 22-47 (1950); these Rev. 12, 290].)
Z. Kopal (Manchester).

Zagar, F. Questioni dinamiche riguardanti gli ammassi stellari sferici. *Rend. Sem. Mat. Fis. Milano* 21 (1950), 28-50 (1951).

The author uses numerical integration to discuss the motion of a star in the potential field for various density distributions in stellar systems with spherical symmetry. The remainder of the paper (relation between projected and spatial distributions in globular clusters, application of Liouville's equation, etc.) is all standard and may be found, for example, in Smart's text [Smart, *Stellar dynamics*, Cambridge, 1938].
R. G. Langebartel (Urbana, Ill.).

***Belzer, Jack, Gamow, George, and Keller, Geoffrey.** Dynamics of elliptical galaxies. *Proceedings, Scientific Computation Forum*, 1948, pp. 67-69. International Business Machines Corp., New York, N. Y., 1950.

The authors set out to analyze Hubble's observed density distribution of stellar population in elliptical galaxies in the light of Gamow's theory of the origin of such systems, which presupposes that their elliptical shape is a remnant from the time when the entire system was in the gaseous state prior to the formation of the individual stars. As this formation took place under the forces of gravity and radiation pressure, the new-born stars will be acted upon by the resistance of the remaining gas, which will diminish in importance as the star formation continues and the gas becomes gradually depleted, and by the attraction of the stars formed at smaller distances from the center. As a result, the stars are expected to oscillate radially through the center of the galaxy (assuming the effects of close encounters to be negligible). The authors set up the equations governing the motions of the individual stars under the conditions just enumerated. Although no solutions are presented, the equations have been reduced to the form which is amenable to numerical solution with the aid of I.B.M. machines.
Z. Kopal (Manchester).

Münch, G., and Chandrasekhar, S. The theory of the fluctuations in brightness of the Milky Way. IV. Astrophys. J. 115, 94-102 (1952).

[For parts I-III see same J. 112, 380-392, 393-398 (1950); 114, 110-122 (1951); these Rev. 12, 644; 13, 249.] In this paper the integral equation governing the brightness of the Milky Way

$$f(u) + \frac{df}{du} = \int_0^1 f(u/q) \psi(q) dq,$$

(where u denotes a measure of the observed brightness of the Milky Way, $f(u)$ the probability that the brightness exceeds the assigned value u , and q is the transparency factor which is assumed to occur with a frequency given by $\psi(q)$) is solved for the case when the system of stars and absorbing clouds extends to infinity in the direction of the line of sight and the transparency factor q characterizing the clouds is governed by an arbitrary frequency function $\psi(q)$ ($0 \leq q \leq 1$). The solution is obtained in the form of an expansion in terms of the Laguerre polynomials, whose coefficients depend only on the moments of $\psi(q)$. Z. Kopal.

Chandrasekhar, S., and Münch, G. The theory of the fluctuations in brightness of the Milky Way. V. Astrophys. J. 115, 103-123 (1952).

In the present paper the authors make an attempt to interpret the observed fluctuations in brightness of the Milky Way on the basis of an assumption that the distribution of density of interstellar matter is continuous, but exhibits fluctuations of a statistical character which are such that the volume absorption coefficient $\kappa\rho(r)$ at any point in the medium can be written as

$$\kappa\rho(r) = \bar{\kappa}\bar{\rho}[1 + \delta(r)],$$

where $\delta(r)$ is a chance variable whose mean expectancy is zero. Furthermore, it is assumed that

$$\overline{\delta^2(r)} = \alpha^2 \quad \text{and} \quad \overline{\delta(r_1)\delta(r_2)} = \alpha^2 R(|r_1 - r_2|),$$

where α is a constant throughout the medium and R represents the correlation coefficient of the fluctuations $\delta(r)$ at

two different points; while the correlation function $R(r)$ defines a micro-scale r_0 such that for $r > r_0$ the correlation rapidly becomes negligible.

Various problems of stellar statistics (such as the fluctuations in the counts of extragalactic nebulae or in the brightness of the Milky Way) are rediscussed in terms of this new picture, and it is shown that most observations can be interpreted equally well in terms of it; moreover, the angular correlations in the observed brightness of the Milky Way in two neighboring directions can be discussed without any additional assumptions. This latter problem is analyzed in the present paper in some detail, and it is shown that Pannekoek's survey of the fluctuations in the brightness of the southern Milky Way leads to a value of $\alpha^2 \sim 14$, which would imply that the root mean square of the deviations in the density is about three to four times the mean density itself. Z. Kopal (Manchester).

Whitrow, G. J., and Randall, D. G. Expanding world-models characterized by a dimensionless invariant. Monthly Not. Roy. Astr. Soc. 111, 455-467 (1951).

A class of model-universes is treated by the methods of kinematical relativity, each model being characterized by the invariance of the dimensionless product $G\rho t^2$, where G is the "constant" of gravitation, ρ is the mean local density of matter and t is the parameter occurring in the red-shift formula. In these models G is proportional to T^{2n-2} where T may be regarded as the age of the model and n is an arbitrary pure number. Thus G is only constant if $n = 2/3$. The parameter t is defined by the hypothesis that the expression for the red-shift displacement of spectral lines is of the form R/at where R is the distance deduced from the apparent magnitude of a galaxy. Different models are obtained by giving different values to n , an increase of age with a concomitant decrease of density being achieved by increasing n from the value $2/3$. The variation of G with the age T becomes increasingly pronounced with increasing n ; thus $n = 3$ gives $G \sim T^2$. For such values of n , the density is of the order of 10^{-28} gr/cm³, whilst the age is about 6×10^9 years. G. C. McVittie (London).

RELATIVITY

Infeld, L., and Dirac, P. A. M. Is there an æther? Nature 169, 702 (1952).

Bondi, H. Relativity and indeterminacy. Nature 169, 660 (1952).

Rumer, Yu. B. Action as a space coordinate. V. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 21, 1403-1411 (1951). (Russian)

In the author's 5-dimensional relativity theory [same Zhurnal 19, 86-94, 207-214 (1949); these Rev. 10, 580] there exists a scalar potential χ in addition to the 10 gravitational potentials g_{ik} and the 4 electromagnetic potentials A_i . Otherwise the field equations and the equations of motion of test-particles are derived in the same way as in Einstein's general theory. In this paper only the classical unquantized form of the author's theory is considered. The question under discussion is, what will be the effect of the χ potential on observable phenomena in the classical theory.

First it is shown that in the Newtonian approximation, and in general whenever the non-linear terms in the field-equations are negligible, the χ potential has no observable

effects. Hence to see the effects it is necessary to solve the field-equations exactly in the only case in which exact solutions of the Einstein equations are known, viz. the static spherically symmetrical case considered by Schwarzschild. The solutions for this case are found very simply, using the field-equations expressed in the form of an action-principle. The 5-dimensional potentials are

$$(1) \quad G_{ik} = \delta_{ik} + (X-1)n_i n_k, \quad G_{44} = G_{55} = 0, \quad i, k = 1, 2, 3; \\ G_{44} = Y + Zg^2, \quad G_{45} = Zg, \quad G_{55} = Z.$$

Here n_i is the unit radius vector and

$$(2) \quad X = (1 - (\gamma/r))^{-1}, \quad Z = 1 + (\alpha/r), \\ Y = (XZ)^{-1}, \quad g = i\epsilon/(r + \alpha).$$

The constants γ, α, ϵ are constants of integration with the following physical interpretation. Let m and e be the mass and charge of the central singularity as determined by the form of the potentials at large distances. The classical gravitational and electrical radii of the singularity are then $\gamma = (\kappa/4\pi)mc^2$, $r_0 = e^2/mc^2$, where κ is the gravitational constant. Finally $c = \sqrt{(2\gamma r_0)}$, $\alpha = \frac{1}{2}(\sqrt{(\gamma^2 + 4r_0^2)} - \gamma)$. These solutions are to be compared with the corresponding solutions

of the Einstein theory including electromagnetic effects, which are given again by (1) but with

$$(3) \quad Y = X^{-1} = 1 - (\gamma/r) + (\epsilon^2/4r^2), \quad Z = 1, \quad g = i\epsilon/r,$$

instead of (2). The difference will only be appreciable when ϵ is not too small, i.e. when the charge ϵ is large. In particular, the angular deflection of a light-ray passing at a distance R from the singularity is given by

$$(4) \quad \Delta\phi = (3\gamma + \sqrt{\gamma^2 + 4\epsilon^2})/(2R)$$

in the author's theory, compared with

$$(5) \quad \Delta\phi = (2\gamma/R) + (3\pi\epsilon^2/16R^2)$$

according to the Einstein theory.

F. J. Dyson.

Thiry, Yves. *Étude mathématique des équations d'une théorie unitaire à quinze variables de champ*. J. Math. Pures Appl. (9) 30, 275-316, 317-396 (1951).

The author discusses a five-dimensional unified theory of relativity. The fifteen components of the metric tensor in the five-dimensional space R_5 are taken to be functions of four variables. They are related to the ten gravitational potentials, the four vector-potentials of the electromagnetic field and a fifteenth function χ which is a generalization of the gravitational constant. A four-dimensional form of the theory presented here has been given by Jordan [Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. 1950, 319-334; these Rev. 13, 79] and others.

The paper is divided into three chapters: The first one is concerned with determining a five-dimensional Riemannian space whose geodesics are identical with the equations of motion for a charged particle in general relativity. The second one is concerned with writing field equations for such a Riemannian space and comparing them with those used in general relativity to describe a gravitational field and an electro-magnetic one. The equations the author chooses have the consequence that an electro-magnetic field is created by a distribution of uncharged stationary matter.

The third chapter is concerned with proving two of the theorems. (1) Throughout a domain of the space R_5 where a distribution of matter is to be introduced the exterior fields created by that distribution must have singularities. (2) If the two unitary fields are asymptotically normal, they cannot be everywhere regular without reducing to zero everywhere. The phrase "asymptotically normal" used above is defined precisely in the paper and involves the behavior of the functions at infinity. These theorems are subject to a number of additional assumptions such as time independence of the fields. There is evidence that corresponding theorems are not true for time dependent solutions of the gravitational field equations [cf. Taub, Ann. of Math. (2) 53, 472-490 (1951); these Rev. 12, 865]. Nevertheless the author argues that the theory presented here is preferable to the unified field theory of O. Klein since theorem (1) does not hold for the latter theory under the assumptions made here. Even this argument is not convincing since it is conceivable that a different choice of the tensor describing matter would change the relation of the theories with respect to these theorems.

A. H. Taub (Urbana, Ill.).

Narlikar, V. V., and Singh, K. P. *The rôle of the three-index symbols in relativity*. Proc. Nat. Inst. Sci. India 17, 311-322 (1951).

Following N. Rosen [Physical Rev. (2) 57, 147-150, 150-153 (1940); these Rev. 1, 183], the authors study spaces with two metric tensors of which one is flat. A theory of

gravitation based on this geometry is developed. The resulting field equations can also be interpreted in non-metric affine geometry.

A. Schild (Pittsburgh, Pa.).

Takeno, Hyôitirô, Ikeda, Mineo, and Abe, Shingo. *On solutions of new field equations of Einstein and those of Schrödinger*. Progress Theoret. Physics 6, 837-848 (1951).

Assuming that the non-symmetric tensor g_{ij} is static and invariant under the group of three-dimensional rotations, the authors obtain solutions to the generalized field equations proposed by Einstein and to those proposed by Schrödinger. The latter differ from the former in that they contain a "cosmological constant". It is shown that the affine connection is not uniquely determined by the g_{ij} under the assumptions made and a new class of solutions of the generalized field equations are obtained.

A. H. Taub (Urbana, Ill.).

el Nadi, Mohamed. *The wave equation in a generalized Riemannian space*. Proc. Math. Phys. Soc. Egypt 4, no. 3, 33-39 (1951). (English. Arabic summary)

G. Randers [Physical Rev. (2) 59, 195-199 (1941); these Rev. 2, 208] and A. Mosharrafa [Philos. Mag. (7) 39, 728-738 (1948); these Rev. 10, 214] have proposed a unified field theory in which space-time is the null cone of a five-dimensional space. The author of the present paper writes a generalization of Dirac's equation for such a field theory. The invariance of the resulting equation under various types of trace formations is not discussed. Nor is a comparison made of the resulting equation with various known generalizations of the Dirac equation which evolve from other five-dimensional unified field theories. It is claimed that certain terms appearing in the author's generalization of the Dirac equation represent radiation interaction.

A. H. Taub (Urbana, Ill.).

Heller, Jack, and Bergmann, Peter G. *A canonical field theory with spinors*. Physical Rev. (2) 84, 665-670 (1951).

Spinors are introduced into relativistic gravitational theory by treating as fundamental field variables the Schrödinger-Dirac matrices γ_μ . The metric tensor can then be defined by $g_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)$. The Lagrangian $2(-g)^{1/2}R$ is expressed in terms of the γ_μ ; this expression is also obtained independently from a spin curvature tensor. The gravitational field equations are written in Hamiltonian form; new equations of constraint appear which are due to the invariance of the theory under spin transformations. Classically this formulation is of course equivalent to general relativity, but the authors feel that quantization may introduce new and distinctive features.

A. Schild.

Hittmair, O., and Schrödinger, E. *Studies in the generalized theory of gravitation. II. The velocity of light*. Communications Dublin. Inst. Advanced Studies. Ser. A. no. 8, i+15 pp. (1951).

[For part I see same Communications no. 6 (1951); these Rev. 12, 757.] The authors study one property of the solutions of the generalized field equations under the following assumptions: (1) The symmetric part of the non-symmetric tensor g_{ij} is Galilean. (2) The anti-symmetric part of g_{ij} is composed of a constant term, the background electromagnetic field and an infinitely weak rapidly oscillating part that represents a light wave whose behavior is to be studied. It is pointed out that under the assumptions made above

the behavior of the light wave is described by the Born-Infeld electrodynamics and the results of that theory are applied. The non-isotropic nature of the velocity of light is then discussed.

A. H. Taub (Urbana, Ill.).

McVittie, G. C. A model universe admitting the interchangeability of stress and mass. Proc. Roy. Soc. London. Ser. A. 211, 295-301 (1952).

The author's summary is as follows: "The possibility of the existence of negative stress in the general relativity treatment of a perfect fluid is used to construct a model universe which is in a 'gravitationally steady state'. Without employing Newtonian analogies, it is shown that stress and mass are mutually convertible into one another in this model, and it is suggested that this process corresponds to the creation of matter postulated in recent cosmological investigations. Using, as the sole empirical datum, the magnitude of the local rate of change of red-shift with distance, plausible assumptions lead to a numerical value of the cosmical constant, to a small value of the density of matter in

space and to an unobservably small rate of conversion of stress into mass. The model has an infinitely long contracting phase, followed by an expanding phase which has been proceeding for at least 9×10^9 years."

A. E. Schild.

Jaiswal, J. P. On the null geodesics and null cones in gravitational fields. Ganita 1, 86-96 (1950).

Explicit expressions are obtained for the null geodesics of the Einstein and de Sitter universes.

A. Schild.

Tonnellat, M. A. Les tentatives de rapprochement entre les constantes λ (constante cosmologique) et μ_0 (masse du photon). J. Phys. Radium (8) 12, 829-832 (1951).

The author points out that by introducing a special coordinate system in a space of constant curvature, the Maxwell equations, when written in terms of the field strength or in terms of the vector potential, have a form similar to the Klein-Gordon equation for a particle with mass. A variety of such mass terms is obtained for various other field quantities.

A. H. Taub (Urbana, Ill.).

MECHANICS

*Artobolevskii, I. I., Bloh, Z., and Dobrovolskii, V. V. Sintez mekhanizmov. [Design of mechanisms.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1944. 387 pp.

A remarkably detailed and comprehensive presentation of one new and two classical methods for the design of cams and linkages. Part I (Artobolevskii, 97 pp.) deals with cams and related mechanisms (Geneva stops, etc.). Part II (Dobrovolskii, 141 pp.) presents the Burmester theory of four-bar linkage design when a number (two, three or four) of corresponding positions of certain members are given, or some similar conditions are prescribed. Part III (Bloh, 122 pp.), the most important one, introduces a method using complex numbers to specify the positions of the hinges, and applies it to eighteen problems of four-bar linkage design and eight problems of five- and six-bar linkage design. The conditions combine a number of relative link positions with angular velocity data, and are more varied than the Burmester ones. Two more volumes, on computing and spatial linkages, are projected.

A. W. Wundheiler.

Easthope, C. E. The existence of a spin integral in the motion of a rigid body in rolling contact with a rough surface. Math. Gaz. 36, 20-29 (1952).

Following a suggestion made by Milne [Vectorial mechanics, Interscience, New York, 1948, p. 357; these Rev. 10, 488] the author proves the theorem: In the general motion of a rigid body spinning and in contact with a second surface, which rotates either about a fixed point or about a fixed axis, the motion being such that there is no slipping at the point of contact, while the forces acting on the system are such that they all intersect the line IG , where I is the point of contact and G is the centre of gravity of the spinning body, an integral of the type $\mathbf{i} \cdot \mathbf{H} = \text{constant}$ (where \mathbf{i} is a unit vector in the direction of IG and \mathbf{H} the angular momentum about G) exists only if: a) IG has a fixed direction in space; b) IG is fixed in the body and coincides with a principal axis of inertia at G ; c) the spinning body is kinetically symmetrical with respect to G and the motion is such that at any instant the velocity of the point of contact relative to the second surface is parallel to $d\mathbf{l}/dt$. Applications to the case that the moving body is a sphere.

O. Bottema (Delft).

Kuz'min, P. A. Supplement to V. A. Steklov's case of motion of a heavy rigid body about a fixed point. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 243-245 (1952). (Russian)

Steklov's case is one of integrability under particular initial conditions. The mass center G lies on a principal axis Ox of the fixed point O , and (*) $Q\gamma_2 = \alpha pq$, $Q\gamma_3 = \beta pr$ where Q is the product of the weight and the distance OG , $\gamma_1, \gamma_2, \gamma_3$ are the cosines of the angles of the vertical with the principal axes at O ; p, q, r are the angular-velocity components along these axes, and α, β two quantities determined by the principal moments of gyration, A, B, C . Eqs. (*) imply $2C < A < 2B$. Steklov dealt in detail with the subcase $A < B$ and showed that p, q, r are proportional to $\text{cn } ht, \text{sn } ht, \text{dn } ht$ where h is determined by A, B, C , and Q . The author carries over this result to the case $A > B$ and obtains the same answer with a different value of h .

A. W. Wundheiler.

Hil'mi, G. F. The evolution of systems of gravitating bodies with inelastic collisions. Doklady Akad. Nauk SSSR (N.S.) 77, 589-592 (1951). (Russian)

In the classical n -body problem let a coordinate system be chosen with origin at the center of mass, and let r_i ($i = 1, \dots, n$) denote the distance from the i th particle, of mass m_i , to the center of mass, so that $I^2 = \sum m_i r_i^2$ is the polar moment of inertia about the center of mass. Let h denote the magnitude of the total angular momentum vector and let T be the total kinetic energy. The inequalities $0 \leq h^2/(2T - \sum m_i \dot{r}_i^2) \leq I^2$ are then established. This result is applied to a qualitative analysis of a system of very many particles which collide inelastically and may, as a result of collisions, either coalesce or split up. It is assumed that the total energy decreases, as a result of collisions. This corresponds to a decrease in T or in the total potential energy. The latter case corresponds to a shrinking of the system and, in particular, coalescing of particles. If the system is assumed bounded (relative to the center of mass), then the above inequalities imply that a continued decrease in T must eventually be accompanied by a decrease in h . This would contradict the constancy of h , unless the total angular momentum is broadened to include rotational angular momentum of the individual particles as rigid bodies; if this is done, the loss in total energy is simply a conversion of

mechanical energy into the "heat" energy of rotation. The results described are interpreted as a possible explanation of the evolution of the solar system from a swarm of meteor-like particles.
W. Kaplan (Ann Arbor, Mich.).

Hil'mi, G. F. On a criterion of indissolubility of capture in the three body problem. Doklady Akad. Nauk SSSR (N.S.) 78, 653-656 (1951). (Russian)

In the classical three-body problem for particles P_0, P_1, P_2 let r_{ij} denote the distance between P_i and P_j ; let r denote the distance between P_2 and the center of mass of P_0, P_1 ; let m_i be the mass of P_i , let $M = m_1 + m_2 + m_0$ and let $b = (m_0 + m_1)m_2 M^{-1}$; let H denote the total energy, the potential energy being 0 when the mutual distances are infinite. It is proved that if the initial conditions are such that $\dot{r} > 0$, $r < 2r_{12}$, $r < 2r_{02}$ and, for some constant R , $r > 2R$, $\dot{r}^2 - 8M^{-1} > 2Hb^{-1} + 2m_0 m_1 b^{-1} R^{-1}$, then the subsequent motion will be such that $r_{01} < R$, while $r \rightarrow +\infty$ as $t \rightarrow +\infty$. This criterion of "capture" is shown to imply one previously established by the author [same Doklady 62, 39-42 (1948); these Rev. 10, 487]. The author remarks that the proof of a third criterion asserted by him [ibid. 71, 1041-1044 (1950); these Rev. 12, 296] has been shown to be defective.

W. Kaplan (Ann Arbor, Mich.).

Hil'mi, G. F. On completely unstable systems of n gravitating bodies. Doklady Akad. Nauk SSSR (N.S.) 79, 419-422 (1951). (Russian)

In the classical n -body problem, let r_{ij} denote the distance between particles P_i and P_j at time t , let

$$g(t) = \min (r_{ij}), \quad s(t) = \min (\dot{r}_{ij}), \quad m_i = \text{mass of } P_i, \\ M' = \sum m_i m_j, \quad M'' = \min m_i m_j / (m_i + m_j), \quad M = M' / M''.$$

Units are chosen so that the constant of gravitation is 1. It is then proved that if $s(0) > 0$ and $s^2(0)g(0) > 8M$, then $g(t) \rightarrow \infty$ as $t \rightarrow \infty$. This criterion for instability is shown to be applicable, by reversal of the time scale, to the problem of "capture" in the three-body problem [cf. the preceding review].
W. Kaplan (Ann Arbor, Mich.).

Platrier, Charles. Contribution à l'étude des actions de la rotation de la terre sur le mouvement local d'un solide. Ann. Ponts Chaussées 121, 655-664 (1951).

The discussion pays particular attention to the terms involving the horizontal component of the earth's rotation.
P. Franklin (Cambridge, Mass.).

Hydrodynamics, Aerodynamics, Acoustics

Shiffman, M., and Spencer, D. C. The force of impact on a cone striking a water surface (vertical entry). Comm. Pure Appl. Math. 4, 379-417 (1951).

This paper treats the exact hydrodynamical problem of vertical entry of a circular cone into a semi-infinite incompressible fluid under the usual assumptions of potential flow, neglect of gravity, and constant pressure on the free surface. The problem for the cone is distinguished by the fact that the flows at every instant are connected by a similarity transformation. This has the effect of reducing the unsteady flow problem to a stationary problem in which the potential ϕ and stream function ψ are to satisfy the conditions $2\phi = s^2 - x^2 - y^2$ and $d\psi = x(ydx - xdy)$ on the unknown free surface; here (x, y) are rectangular coordinates in the meridian plane with the origin at the point where the vertex

first touches the free surface, and s is the arclength along the free surface from the topmost wetted point of the cone to the point (x, y) . Explicit solution being apparently out of the question, the authors fall back on a complex iteration procedure for computing the potential and free surface from a non-linear singular integral equation derived by means of Green's Theorem. The authors outline the iteration scheme and present the results of computations, in particular the impact force, for a cone of vertex angle 120° , as carried out by A. Hillman in an unpublished report. Since the calculations based on the exact theory are formidable, the authors present a simpler approximate theory which compares well with the exact calculations for the 120° cone and seems to compare favorably with several experimental results of S. Watanabe on cones of large vertex angle. The analogous two-dimensional problem of vertical impact of a wedge has been considered by H. Wagner [Z. Angew. Math. Mech. 12, 193-215 (1932)]. The basic theory is essentially the same in the two cases but the treatments are different and the computational details are lacking in Wagner's paper.

D. Gilbarg (Bloomington, Ind.).

Peters, Arthur S. Water waves over sloping beaches and the solution of a mixed boundary value problem for $\Delta^2 \phi - k^2 \phi = 0$ in a sector. Comm. Pure Appl. Math. 5, 87-108 (1952).

L'auteur étudie le problème de la houle progressive sur un rivage plan incliné, lorsque la direction de propagation est quelconque. La condition sur la surface étant linéarisée, ce problème revient à la détermination dans l'angle $-\gamma < \theta < 0$, $0 < \rho < +\infty$, d'une solution $\phi(\rho, \theta)$ de l'équation $\rho^2 \phi_{\rho\rho} + \rho \phi_\rho + \phi_{\theta\theta} - \rho^2 k^2 \phi = 0$ assujettie aux conditions limites $\phi_\theta - \rho \phi = 0$ sur $\theta = 0$, $\phi_\theta = 0$ sur $\theta = -\gamma$; la constante k désignant l'angle de la direction de propagation avec la normale au rivage. La méthode de l'auteur repose sur l'utilisation de la transformation de Laplace qui permet de transformer ce problème en celui de la détermination d'une fonction harmonique dans une bande indéfinie et qui vérifie sur la frontière des relations linéaires par rapport à ses dérivées premières. Il est alors possible de construire sous forme d'intégrales des solutions explicites de ce problème mixte et d'étudier l'allure de ces solutions à l'origine. Signalons que les méthodes de l'auteur vaudraient pour l'étude de l'équation $\Delta^2 \phi - k^2 \phi = 0$ lorsque ϕ est définie dans un angle et satisfait sur la frontière à des conditions du type envisagé.

R. Gerber (Grenoble).

Storchi, Edoardo. Legame fra la forma del pelo libero e quella del recipiente nelle oscillazioni di un liquido. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 13(82), 95-112 (1949).

Assuming 2-dimensional motion of an inviscid incompressible heavy fluid and the linearized boundary condition for gravity waves, the author finds that if the free surface has the form $\zeta(t, x) = \rho'(x)\psi(t)$, then $\psi(t)$ is sinusoidal with frequency ω and the streamlines (including the shape of the container) are given by

$$\omega^2 g^{-1} [\rho(x+iy) + \rho(x-iy)] + i[\rho'(x+iy) - \rho'(x-iy)] = K,$$

where K is a real constant and the left-hand side must be real for the chosen function $\rho(x)$ to be possible. A number of special choices of $\rho'(x)$ are treated, giving both new and old results.

J. V. Wehausen (Providence, R. I.).

Martin, J. C., Moyce, W. J., Penney, W. G., Price, A. T., and Thornhill, C. K. Some gravity wave problems in the motion of perfect liquids. *Philos. Trans. Roy. Soc. London. Ser. A.* 244, 231-334 (1952).

This paper is divided into five separate parts of which the first three will be treated separately. There is a "General Introduction" by Penney outlining the papers and describing their genesis.

Part I: The diffraction theory of sea waves and the shelter afforded by breakwaters (pp. 236-253) by Penney and Price. Using the linearized water wave theory, the authors consider sinusoidal waves moving toward a half-infinite breakwater, $y=0$, $x \geq 0$, where $x=0$ is the undisturbed water surface. Two boundary conditions on the breakwater are considered: (1) $\varphi_y=0$, the "rigid" breakwater, and (2) $\varphi_1=0$, the "cushion-type" breakwater. It is shown that the corresponding two problems are the same as two problems in light diffraction by a semi-infinite screen, the two boundary conditions corresponding to the polarization of the light. Using the known solution for this case they discuss the resulting wave pattern for waves moving both normally and obliquely toward the breakwater. The solutions are used to obtain approximate solutions for a breakwater of finite length and two half-infinite breakwaters separated by a gap.

Part II: Finite periodic stationary gravity waves in a perfect liquid (pp. 254-284) by Penney and Price. Let Ox be in the mean water level, Oy vertically upwards and let $f(x, y, t)=0$ represent the free surface of the water. Considering first 2-dimensional motion in water of infinite depth, the authors seek a solution of $\varphi_{xx} + \varphi_{yy} = 0$, $\varphi_x(0, y, t) = 0$, $\lim_{y \rightarrow \infty} \varphi_y(x, y, t) = 0$, and

$$-gy + \varphi_t - \frac{1}{2}(\varphi_x^2 + \varphi_y^2) = 0 \quad \text{if} \quad f(x, y, t) = 0.$$

They look for a solution in the form $\varphi = \sum_0^\infty \alpha_n(t) e^{nky} \cos nkx$ where φ will also be periodic in t . Also it is assumed that $f(x, y, t)=0$ may be written in the form $y = \sum_0^\infty a_n(t) \cos nkx$, and that as the amplitude becomes small the free surface approaches the form $y = C \sin(kg)t \cos kx$. By assuming the existence of such a solution, they are able to determine successively the α_n and a_n . The calculations are carried out up to $n=5$, and the corresponding ones for water of finite depth up to $n=2$. [Sekerž-Zen'kovič [Doklady Akad. Nauk. SSSR (N.S.) 58, 551-553 (1947); *Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz.* 15, 57-73 (1951); these *Rev.* 10, 646; 12, 870] has carried out similar computations by a quite different method up to $n=4$ and $n=3$ in the two cases. Although the general form of the solutions and the qualitative conclusions agree, there seems to be some discrepancy in numerical coefficients.] By imposing the additional condition, $\partial p / \partial y \leq 0$ on the free surface, (which they support by several arguments) they show that there is an upper bound to the ratio amplitude/length of the waves. For this greatest wave (if it exists) the condition $\partial p / \partial y = 0$ at the crest leads to the conclusion that the crest is pointed, the two sides making an angle of 90° . Other related questions are also treated.

Part III: The dispersion, under gravity, of a column of fluid supported on a rigid horizontal plane (pp. 285-311) by Penney and Thornhill. The authors consider a humped up portion of a fluid of density ρ_1 resting on a plane and surrounded by a fluid of smaller density ρ_2 . The hump is suddenly allowed to disperse under the action of gravity and the subsequent development of its shape is studied under the assumption of perfect incompressible fluids and either plane or axial symmetry. Three approximate methods of solving

the resulting equations are given and applied to special cross-section shapes for the initial hump. An appendix by L. Fox and E. T. Goodwin describes a fourth approximate method of solution. Parts IV and V are experimental studies by Martin and Moyce of the same problem.

J. V. Wehausen (Providence, R. I.).

Coddington, E. A. The stability of infinite differential systems associated with vortex streets. *J. Math. Physics* 30, 171-199 (1952).

This paper marks the first time that the stability of the vortex street has been treated in terms of the full infinite system of equations for the small displacements. Before entering into the hydrodynamical stability problem, the author considers a class of infinite systems of differential equations of the form $(*) \dot{\xi} = A\xi$, where A is an infinite constant matrix with certain prescribed properties, and $\xi = \xi(t) = (\xi_n(t))$, $n=0, \pm 1, \pm 2, \dots$, is an infinite vector with k -dimensional vectors $\xi_n = (\xi_n^i)$, $i=1, \dots, k$, as elements. After constructing explicitly the solution of the initial value problem for $(*)$, he then establishes stability criteria for the solutions, formulated in terms of the Hilbert norm $\|\xi\| = (\sum_{n=-\infty}^{\infty} \sum_{i=1}^k |\xi_n^i|^2)^{1/2}$, these criteria being quite analogous to those known for finite systems. For a vortex street the infinite system of linearized equations obeyed by the small displacements from equilibrium belongs to the class of equations $(*)$ with $k=4$. The author is thus able to draw conclusions concerning stability of the vortex street, where stability is of course understood with respect to initial displacement vectors of finite norm. For a street with vortices situated (in equilibrium) at the points $(2nb+c, a)$, $(2nb-c, -a)$, $n=0, \pm 1, \pm 2, \dots$, he shows that if $q=c/b \neq \frac{1}{2}$ there are initial displacement vectors of finite norm for which $\|\xi(t)\|$ becomes infinite exponentially as $t \rightarrow \infty$, so that the street is unstable. This is of course in agreement with von Kármán's classical result. For the case $q=\frac{1}{2}$ a necessary condition for stability is $(**) \cosh^2 \pi a/b = 2$, again in agreement with the classical theory. However, the author shows by counterexample that even when this condition is satisfied there are initial displacement vectors for which $\|\xi(t)\| \rightarrow \infty$ faster than t^β as $t \rightarrow \infty$ for any β , $0 < \beta < 2$. This contradicts the well-known stability criterion for the Kármán street which, we recall [Lamb, *Hydrodynamics*, 6th ed., Cambridge, 1932, pp. 225-229], is proved in a different sense from that of the present paper and for only a restricted class of allowed initial displacements (which in addition have infinite norm and are therefore not admitted in the present discussion). The author shows further, as have several authors [see, e.g., Goldstein, *Modern Developments in Fluid Dynamics*, v. 2, Oxford, 1938, pp. 563-564], that even in Kármán's sense the vortex street is unstable with respect to arbitrary initial disturbances. However, he displays a wide class of finitely normed initial displacements for which the Kármán criterion $(**)$ is sufficient, as well as necessary, for stability. The paper concludes with a detailed discussion of the stability of the double vortex street. D. Gilbarg.

Braun, I., and Reiner, M. Problems of cross-viscosity. *Quart. J. Mech. Appl. Math.* 5, 42-53 (1952).

The second author has shown [*Amer. J. Math.* 67, 350-362 (1945); these *Rev.* 7, 44] that if Stokes's definition of a fluid as a continuum in which the excess of the pressure tensor p_{ij} above that corresponding to equilibrium at the same temperature and pressure is a function of the rate of deformation tensor f_{ij} only be adopted, then for isotropic

fluids we have

$$p_j^i = F_0 \delta_j^i + F_1 f_j^i + F_2 f_j^i f_j^j,$$

where F_0 , F_1 , and F_2 are functions of material constants, thermodynamic state, and the principal invariants of f_j^i . In the present paper the authors obtain solutions of the dynamical equation appropriate to Couette flow, simple Poiseuille flow, and the motion in a parallel plate viscometer. They take F_1 and F_2 as constants, thus obtaining results valid only to the second order in f_j^i ; they neglect the inertia of the fluid. While they mention Rivlin's discussion [Proc. Roy. Soc. London. Ser. A. 193, 260-281 (1948); these Rev. 10, 73] of one of the phenomena they discuss, they appear to have overlooked the fact that for an incompressible fluid, where F_0 is an arbitrary hydrostatic pressure, all their results have been obtained previously by him for fluids with arbitrary F_1 and F_2 [loc. cit. and Proc. Cambridge Philos. Soc. 45, 88-91 (1949); these Rev. 10, 214]. The reviewer notes that for compressible fluids the authors' results are not truly solutions, since they take no account of the energy equation and the equation of state. C. Truesdell.

Wolbner, W. Sur le mouvement plan du liquide visqueux, incompressible, entourant une courbe simple fermée. *Studia Math.* 12, 279-285 (1951).

Let C be a closed curve with continuous tangent, forming the interior boundary of a viscous incompressible fluid moving in a plane. Relative to some Newtonian frame N , C has unit velocity in a fixed direction, and the author is interested in motions of the fluid which are steady relative to C and of finite kinetic energy E relative to N . He measures velocity (u, v) relative to N , but regards the components as functions of coordinates (x, y) relative to C . [It seems to the reviewer that it would be simpler to use C as frame of reference for both velocity and coordinates and express the energy condition by saying that $\int \int [(u+1)^2 + v^2] d\sigma$ is finite.] Transforming to polar coordinates (r, ϕ) , he obtains expressions for the components of force P_x , P_y on C ; that for P_x is

$$(*) \quad P_x = - \int_{H_r} \int \frac{\partial u}{\partial t} d\sigma - r^2 \int_0^{2\pi} \frac{\partial u_\phi}{\partial t} \sin \phi d\phi + F(r),$$

where H_r is the region between C and a circle K_r of any radius r , and $F(r)$ involves only integrals on K_r . Using the finiteness of E and Schwarz's inequality, he proves the existence of an infinite sequence of radii $\{r_n\}$ such that $\lim_{n \rightarrow \infty} F(r_n) = 0$. On making the further hypothesis that radial velocity and pressure are bounded, he proves that

$$(**) \quad P_x = - \lim_{n \rightarrow \infty} (D_{r_n}'' + \partial E_{r_n}'' / \partial t),$$

where $\{r_n''\}$ is an infinite increasing sequence and

$$D_r = R^{-1} \int_{H_r} \int \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] d\sigma,$$

$$E_r = \frac{1}{2} \int_{H_r} \int (u^2 + v^2) d\sigma$$

(R is the Reynolds number). He concludes from (*) that in a steady motion no force is exerted on C . Further, from (**), under the boundedness conditions on velocity and pressure, he concludes that no steady motion exists satisfying the requirements. This conclusion is similar to that of P.

Udeschini [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 957-963 (1941); these Rev. 8, 419], who however used a boundary condition at infinity instead of the condition of finite energy. J. L. Synge (Dublin).

Rumer, Yu. B. The problem of a submerged jet. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 16, 255-256 (1952). (Russian)

The author considers solutions of the Navier-Stokes equations in spherical coordinates which are regular in θ and of the following form:

$$v_r = r^{-1} F_1(\theta) + r^{-2} F_2(\theta), \quad v_\theta = r^{-1} f_1(\theta) + r^{-2} f_2(\theta), \\ v_\phi = 0, \quad p/\rho = r^{-1} g_1(\theta) + r^{-2} g_2(\theta).$$

The functions F_1 , f_1 and g_1 were determined earlier by Yacev [Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 20, 1031-1034 (1950); these Rev. 12, 552]. The author determines F_2 , f_2 and g_2 corresponding to a particular choice of the F_1 , f_1 and g_1 , namely that resulting when $a=b=c=0$ (see the cited review for notation). This may be interpreted as the flow resulting when a pipe discharges into a space filled with the same fluid. J. V. Wehausen.

Nevzglyadov, V. G. On the boundary conditions of a new method in the dynamics of viscous fluids. *Doklady Akad. Nauk SSSR (N.S.)* 82, 213-216 (1952). (Russian)

The author elaborates further his method of approximate solution of the Navier-Stokes equations [same Doklady 77, 573-576, 795-798 (1951); these Rev. 13, 82, 83]. In particular, he wishes to specify further the polynomial approximation to u within the boundary layer mentioned in the review of the first cited paper. For this he requires the polynomials to satisfy the Navier-Stokes equations. Application is made to the problem of the second cited paper and a number of further approximations are made.

J. V. Wehausen (Providence, R. I.).

Kalinin, N. K. Filtration through a double-layered wedge. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 16, 213-222 (1952). (Russian)

A dam of wedge-shaped cross-section consists of two types of media with different permeabilities. These media are separated by a plane which contains the ridge of the dam. It is required to find the directions and speeds of the flow of the water (which is filtrating through the dam) in each layer of the dam. The problem consists in the determination of two functions of a complex variable, which satisfy certain boundary conditions and each of which is regular within one of two angular regions with a common boundary. The solution is sought and found in the form of Fourier integrals. H. P. Thielman (Ames, Iowa).

Meksyn, D. Numerical integration of the boundary-layer equation. *Proc. Roy. Soc. London. Ser. A.* 209, 375-379 (1951).

A method of step-by-step integration for the development of the boundary layer is applied to the case of the Schubauer ellipse, but the integration was not carried out to give the point of separation. [The author states that Schubauer's observed separation point occurs for the value 0.310 of the parameter λ used in the theory, yet calculated points are given for $\lambda=0.20$ and $\lambda=0.32$ without separation being obtained.] C. C. Lin (Cambridge, Mass.).

Meksyn, D., and Stuart, J. T. Stability of viscous motion between parallel planes for finite disturbances. *Proc. Roy. Soc. London. Ser. A.* 208, 517-526 (1951).

The authors consider a linearized periodic oscillation about a profile slightly distorted by the Reynolds stress of the oscillation. In this way, non-linear effect is partly taken into account. A critical Reynolds number of 2900 is found. *C. C. Lin* (Cambridge, Mass.).

Chandrasekhar, S. The gravitational instability of an infinite homogeneous turbulent medium. *Proc. Roy. Soc. London. Ser. A.* 210, 26-29 (1951).

Jeans's analysis of the problem of gravitational stability of an infinite homogeneous medium is extended to include the effects of turbulence, and it is shown that the eddies in the density fluctuations of wave numbers

$$k < [4\pi G \bar{\rho} / (c^2 + \frac{1}{2} \bar{u}^2)]^{1/2},$$

where $\bar{\rho}$ denotes the density of the medium, c the velocity of sound in this medium, and \bar{u}^2 the mean square velocity of turbulence, are unstable in the sense that these fluctuations will grow with time. *Z. Kopal* (Manchester).

Pai, S. I. On the stability of two-dimensional laminar jet flow of gas. *J. Aeronaut. Sci.* 18, 731-742 (1951).

The author studies the stability of a laminar jet of gas, for both low and high Mach numbers. When compressibility can be neglected, the general stability characteristics are similar to those for the mixing zone studied by Chiarulli and Lessen. The main difference lies in the existence of the symmetrical and the antisymmetrical modes of oscillation, both of which are unstable at large Reynolds numbers. In the compressible case, the effect of varying Mach number is discussed by means of some approximation formulae. It is concluded that the symmetrical disturbances become more stable as the Mach number is increased, and become completely stable when the Mach number reaches a critical value beyond which no subsonic disturbances are possible. On the other hand, the stability characteristics of the antisymmetrical disturbance have a rather complicated dependence on the Mach number. *C. C. Lin*.

Mattioli, Ennio. Le relazioni tra le funzioni di correlazione della velocità nella turbolenza omogenea e isotropica. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) 11, 260-264 (1951).

Derivations of the known relations between the correlation functions f, g, h, k, q of von Kármán and Howarth [Proc. Roy. Soc. London. Ser. A. 164, 192-215 (1938)].

J. V. Wehausen (Providence, R. I.).

Sen, N. R. On Heisenberg's spectrum of turbulence. *Bull. Calcutta Math. Soc.* 43, 1-7 (1951).

The author studies the equation of spectrum of turbulence with Heisenberg's form of the transfer function. Self-preserving solutions are discussed for large Reynolds numbers and small Reynolds numbers, giving essentially results known before. A partial differential equation using Chandrasekhar's variables is derived. This is again used to derive the same results. *C. C. Lin* (Cambridge, Mass.).

Carafoli, E., et Oroveanu, T. L'aile traversant un jet de section circulaire. *Acad. Repub. Pop. Române. Bul. Ști. A.* 1, 695-715 (1949). (Romanian, Russian, and French)

Consider a wing of span b at incidence α which completely spans an open circular jet of diameter b with velocity V_0 at

infinity and with its axis in the median plane of the wing. Let $\Gamma = \Gamma(y) = 2bV_0 \sum A_n \sin n\theta$ be the circulation at distance $y = -\frac{1}{2}b \cos \theta$ from the median plane. To satisfy the boundary condition at the surface of the jet, for the elementary vortex of strength $d\Gamma$ at y , a corresponding image vortex of strength $d\Gamma$ must be introduced at $b^2/4y$. Then A_n can be determined from (1) $\Gamma = a_0 c(V_0 \alpha - w - w_0)$, where a_0 is a constant, $c = c(y)$ is the chord, the induced velocity

$$w = (4\pi)^{-1} \int_{-b/2}^{b/2} (\eta - y)^{-1} (d\Gamma/d\eta) d\eta,$$

and the additional induced velocity due to the image system, $w_0 = (4\pi)^{-1} \int_{-b/2}^{b/2} (\eta - b^2/4y)^{-1} (d\Gamma/d\eta) d\eta$. To simplify the computations of A_n the authors approximate w , as follows. Since the wing's trailing vortex sheet curls up at the edges about cores inboard of the wing tips, replace it by a horseshoe vortex of strength $\Gamma_0 = \Gamma(0)$ of span κb , determined by requiring $\Gamma_0 \kappa b = \int_{-b/2}^{b/2} \Gamma(y) dy$. Replace the image system by vortices of strength $\pm \Gamma_0$ at $\pm b^2/2\kappa b$, for which

$$w_0 = (V_0 A_1 / 2\kappa^2) / (\kappa^2 - \cos^2 \theta).$$

For this w , and $c_0 \sin \theta / c(y) = \beta_0 + 2\beta_2 \cos 2\theta + 2\beta_4 \cos 4\theta$ (rectangular or trapezoidal wings) (1) yields recursion formulas for A_n containing the parameter $\kappa = \frac{1}{2} A_1 / \sum (-1)^n A_{2n+1}$. To obtain κ the terms $n \geq 3$ are neglected, and other simplifications are made in the equations for A_3 and A_4 . As usual the lift coefficient $c_L = \pi \lambda A_1$, where λ is the wing's aspect ratio, and the induced drag coefficient $c_{di} = (c_L^2 / \pi \lambda) \times$ (a known series in A_{2n+1}). *J. H. Giese*.

de Schwarz, Maria Josepha. Oscillazioni armoniche di ali triangolari con bordo d'attacco supersonico. *Aerotecnica* 31, 288-298, 306 (1951).

J. W. Miles [*J. Aeronaut. Sci.* 16, 568-569 (1949); these Rev. 11, 273] has exhibited simple integrals for the lift and rolling moment of supersonic delta wings performing harmonic plunging, pitching, and rolling oscillations. The author gives a detailed derivation of these equations and also computes the pitching moment by interchanging the order of integration of a quadruple integral and differentiating with respect to the wing chord, l , as suggested by the work of A. H. Flax [*ibid.* 16, 496-504 (1949); these Rev. 11, 224] on reversed flow. The lift and moment coefficients are expressed as linear combinations of Bessel functions $J_0(\omega, \sin \theta)$, $J_1(\omega, \sin \theta)$, and of the functions

$$-J_0(\sin \theta, \omega_r) + iJ_1(\sin \theta, \omega_r) = \int_0^{\omega_r} J_0(u \sin \theta) e^{iu} du,$$

tabulated by L. Schwarz [*Luftfahrtforschung* 20, 341-372 (1944); these Rev. 5, 238], where θ is the Mach angle of the undisturbed flow at velocity U , $\omega_r = \omega l / U \cos^2 \theta$, and ω is the frequency. Lift and moment coefficients have been computed at Mach numbers 1.25 and 2 for $0.02 \leq \omega_r \leq 2.0$. The limiting forms of the pressure distributions at these Mach numbers for sonic leading edges are also shown. The author notes the similarity between her work and part of that of Froelich [*J. Aeronaut. Sci.* 18, 298-310 (1951); these Rev. 13, 86], even to the same choice of numerical examples. *J. H. Giese* (Havre de Grace, Md.).

Delval, J. Sur la dynamique des fluides parfaits et le principe d'Hamilton. *Acad. Roy. Belgique. Bull. Cl. Sci.* (5) 37, 986-990 (1951).

Basset [A treatise on hydrodynamics, v. 1, Deighton Bell, Cambridge, 1888, see §4] derived Euler's dynamical equation for ideal compressible fluids from a Hamiltonian varia-

tional principle, using the continuity equation as a side condition. The author prescribes the Eulerian coordinates $x^i = \xi^i$ of the boundary points of a given fluid mass. By adding to the action integral the quantity

$$\int_0 \oint \mu_i (x^i - \xi^i) d\sigma dt,$$

where σ is the Lagrangian boundary surface and the components μ^i are multipliers, he thus obtains boundary conditions as well. *C. Truesdell (Bloomington, Ind.).*

Longhorn, A. L. The unsteady, subsonic motion of a sphere in a compressible inviscid fluid. *Quart. J. Mech. Appl. Math.* 5, 64-81 (1952).

This paper deals with two problems: (a) the motion produced by a sphere set impulsively into uniform motion and (b) that produced by a sphere accelerated gradually from rest to a certain velocity which is then maintained constant. By the acoustic approximation, in case (a) it is shown that the velocity potential approaches the steady value exponentially as time increases but the discontinuity in normal velocity on the wave front attenuates as the inverse distance from the center of the sphere. The force required to maintain the motion and the work done by this force are calculated; and it is interesting to note that the energy ultimately imparted to the fluid is twice as much as when the compressibility is neglected.

For case (b), both the velocity potential and the work done are expressed, in integral forms, in terms of the given variable velocity of the sphere in a finite time interval. When the acceleration is uniform, the work done in time t can be evaluated. It is shown that the virtual mass approaches the incompressible value if the acceleration is very gradual over a long time interval; but if the acceleration is very steep for a short time interval, the limiting value will be double that of the incompressible case, as in problem (a). Finally, an improved equation for the velocity potential is proposed and a possible way of solution is indicated.

Y. H. Kuo (Ithaca, N. Y.).

Germain, Paul, et Fenain, Maurice. Sur une correspondance simple entre les solutions de deux équations aux dérivées partielles, et son application à l'étude approchée des écoulements transsoniques. *C. R. Acad. Sci. Paris* 234, 592-594 (1952).

The authors give a method which transforms an approximate differential equation for the isentropic flow of gases in hodograph variables to the well-known Tricomi equation. One specific example for a nozzle flow is given. From a more general point of view, C. Loewner [*Tech. Notes Nat. Adv. Comm. Aeronaut.*, no. 2065 (1950); these *Rev.* 13, 464] has previously treated this same problem. *Y. H. Kuo.*

Riabouchinsky, Dimitri. Sur les singularités du régime transsonique et le problème du profil de résistance minima aux vitesses supersoniques. *C. R. Acad. Sci. Paris* 233, 1330-1333 (1951).

The author writes the linearized equations of small perturbations of a stream, for both subsonic and supersonic cases. After choosing solutions of a special form, he reaches the incredible conclusions (i) that for subsonic flow the pressure at the boundary of a cylinder is proportional to $|dy/dx|$, where $y(x)$ is the cylinder contour, and (ii) that for supersonic flow it is proportional to dy/dx as in Ackeret's well-known approximation, but with an arbitrary coefficient

of proportionality. By proper choice of this constant, he believes that he can avoid the singularity at sonic stream speed. *W. R. Sears (Ithaca, N. Y.).*

Ferrari, Carlo. On rotational conical flow. *Tech. Memos. Nat. Adv. Comm. Aeronaut.*, no. 1333, 12 pp. (1952). Translated from *Aerotecnica* 31, 64-66 (1951); these *Rev.* 13, 295.

Denisse, Jean François, et Rocard, Yves. Excitation d'oscillations électroniques dans une onde de choc. Applications radioastronomiques. *J. Phys. Radium* (8) 12, 893-899 (1951).

The authors of this investigation study the effects accompanying the propagation of a shock wave through a highly ionized gas. They show that, in general, the diffusion will tend to drive the electrons in front of the shock. As a result, the medium becomes polarized (which eventually sets an upper limit to the effects which the diffusion can produce), and the velocity-distribution of the electrons tends to deviate greatly from a maxwellian one. In point of fact, for sufficiently strong shock waves the electrons tend to split up into two groups, each moving with a different mean velocity. This should give rise to the plasma oscillation, and the authors suggest that a mechanism of this nature may possibly be responsible for the observed occasional outbursts of the solar radiation in the meter waves of its radio spectrum. *Z. Kopal (Manchester).*

Bechert, Karl, und Marx, Helmut. Ebene Wellen endlicher Amplitude in idealen Gasen. *Z. Naturforschung* 6a, 767-775 (1951).

Kofink, W. Berichtigung zur Arbeit "Zur Theorie des gegabelten Verdichtungsstosses". *Ann. Physik* (6) 10, 200 (1952).

See *Ann. Physik* (6) 9, 200-212 (1951); these *Rev.* 13, 296.

Owen, P. L., and Thornhill, C. K. The flow in an axially-symmetric supersonic jet from a nearly-sonic orifice into a vacuum. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2616 (11,768), 8 pp. (1952).

Trilling, Leon. The collapse and rebound of a gas bubble. *J. Appl. Phys.* 23, 14-17 (1952).

A spherical gas bubble in an infinite liquid collapses under the influence of superior external pressure. The liquid is assumed slightly compressible, to the extent of supporting sound propagation, and the gas is considered compressible; finally, adiabatic. It is assumed that flow velocities are small compared with the velocity of sound in the liquid, and that all shocks are weak. The external acoustic theory is given in detail, but the internal adiabatic (approximate) theory is outlined only. The progress of the bubble radius and the shock waves as functions of time is shown graphically. The pressure distribution on the bubble surface as a function of the radius is also given. An interesting numerical example is given showing that a surprisingly high maximum pressure may be obtained in such a collapse. *E. Pinney.*

Van Mieghem, J. Le bilan de la rotationnelle absolue dans l'atmosphère. *Tellus* 3, 297-300 (1952).

The author derives a differential equation for the rate of change of absolute vorticity as apparent to an observer in an arbitrary reference frame, with respect to which the fluid

velocity is v . The result,

$$\frac{\partial \xi}{\partial t} + \text{div}(\xi v - v\xi) = \text{curl } a,$$

where a is the acceleration, might have been obtained by purely kinematic means [cf. G. Jaffé, *Phys. Z.* 22, 180-183 (1921); the reviewer, *Physical Rev.* (2) 73, 510-512 (1948); these *Rev.* 9, 474]. The author calls the term ξv "convective flux," the term $v\xi$ "non-convective flux" [cf. the reviewer, loc. cit. and *C. R. Acad. Sci. Paris* 227, 757-759, 821-823 (1948); these *Rev.* 10, 490; 11, 221]. He uses the results to obtain and interpret equations governing the rate of change of a meteorological observable. A similar device was used by the reviewer to derive the Bjerknes theorem [Three lectures on mathematics and mechanics, Naval Research Lab. Theor. Mech. Sect., Mem. no. 3836-1, pp. 21-37 (1949)] and the Ertel theorem [*Z. Angew. Math. Physik* 2, 109-114 (1951); these *Rev.* 12, 761; cf. also Ertel and Köhler, *Z. Angew. Math. Mech.* 29, 109-113 (1949); these *Rev.* 11, 62].

C. Truesdell (Bloomington, Ind.).

*Schoch, Arnold. *Schallreflexion, Schallbrechung und Schallbeugung*. Ergebnisse der exakten Naturwissenschaften, Band 23, pp. 127-234. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1950.

This monograph is a valuable addition to the literature of the theory of sound waves of small amplitude, either in a gas or in an elastic solid. The scope of the work is best indicated by the titles of the chapters, viz.: (I) Introduction; (II) Foundations of the theory; (III) Reflexion and refraction of a plane wave at a plane boundary surface; (IV) Free boundary-layer waves along a plane boundary surface; (V) Reflexion and refraction of non-plane waves at a plane boundary surface; (VI) Waves in plates; (VII) Layered media; (VIII) Curved boundary surfaces and diffraction phenomena; (IX) Bibliography. The bibliography should prove most useful, as it contains 144 references, nearly half of them to work done since 1945. Whilst the work is essentially mathematical, it contains many beautiful photographs of experiments illustrating the theory. E. T. Copson.

Tartakovskii, B. D. On the passage of sound waves through the boundary of solid and liquid media. *Akad. Nauk SSSR. Zhurnal Tehn. Fiz.* 21, 1194-1201 (1951). (Russian)

Vengono discusse le note formule per il passaggio di onde piane attraverso alla superficie piana che separa due diversi mezzi elastici, nel caso particolare che si tratti di un liquido e di un solido; per semplificare la notazione viene introdotta una resistenza di entrata, relativa alla superficie di separazione.

G. Toraldo di Francia (Firenze).

Stevenson, A. F. Exact and approximate equations for wave propagation in acoustic horns. *J. Appl. Phys.* 22, 1461-1463 (1951).

In order to eliminate the errors involved in the application of previous approximate formulas regarding wave propagation in acoustic horns, the author proposes a more exact theory in which the assumptions that the wave fronts are plane, are dropped. The solution of the wave equation is developed in an infinite series of eigenfunctions relating to each cross-section. Upon determination of the constants, this general solution is discussed in the case that coupling between modes is neglected. M. J. O. Strutt (Zurich).

Elasticity, Plasticity

Truesdell, C. The mechanical foundations of elasticity and fluid dynamics. *J. Rational Mech. Anal.* 1, 125-171, 173-300 (1952).

L'autore fa una chiara ed organica esposizione delle ricerche, più o meno recenti, miranti a sostituire l'ordinaria teoria dell'Elasticità e l'ordinaria Dinamica dei fluidi viscosi con teorie più aderenti all'effettivo comportamento dei corpi reali. Trattasi di un'ampia relazione corredata da una ricca bibliografia (più di 700 citazioni relative ad epoche variabili dal 1676 ai nostri giorni). Particolarmente interessante è il capitolo dedicato alla teoria dell'Elasticità in cui l'autore riferisce sui più recenti risultati (Rivlin, Signorini, ecc.) conseguiti in merito al problema della ricerca di soluzioni esatte delle equazioni dell'Elastostatica nella forma che ad esse compete quando si considerino deformazioni finite e si esca quindi dal campo di validità dell'ordinaria legge di Hooke-Cauchy. Nel capitolo dedicato alla Dinamica dei fluidi l'autore espone alcune interessanti generalizzazioni (Rivlin, Truesdell) della legge di Newton-Cauchy-Poisson su cui si fonda la teoria classica dei fluidi viscosi isotropi. Un capitolo è dedicato pure alla teoria dei corpi il cui comportamento è intermedio tra quello dei corpi elastici e quello dei fluidi viscosi. Completa il lavoro l'enunciazione di un programma di future ricerche da compiere a completamento di quelle su cui l'autore ha riferito. C. Tolotti (Napoli).

Truesdell, Clifford A. A program of physical research in classical mechanics. *Z. Angew. Math. Physik* 3, 79-95 (1952).

A simplified account of some of the ideas treated in detail in the paper reviewed above, a discussion of inadequacies of the linear theories of elasticity and viscosity, and a list of suggested investigations, both theoretical and experimental, in the nonlinear theories.

Angles d'Auriac, Paul. Sur une forme géométrique des conditions d'équilibre des surfaces déformables. *C. R. Acad. Sci. Paris* 234, 294-295 (1952).

Let a variable interior point M of a simply-connected elastic surface be connected to a fixed point A of the boundary by a curve AM ; set at M the initial points of the vectors of resultant force and moment exerted by one side of the curve upon the other; let μ and ν be the termini of these vectors. Then μ and ν are independent of the path AM , and as M traverses S , μ and ν traverse image surfaces Σ and Σ_1 . For the case of an infinitely flexible surface, the author states that the tangent planes at corresponding points of S and Σ are parallel; that there exists a net of orthogonal isostatic curves on S ; and that Σ is a possible equilibrium configuration of S . Explanations, proofs, and further results are to appear elsewhere. C. Truesdell.

Filonenko-Borodich, M. M. Two problems on the equilibrium of an elastic parallelepiped. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 15, 563-574 (1951). (Russian)

This paper is a detailed presentation of an earlier one of the author [same journal 15, 137-148 (1951); these *Rev.* 13, 92]. The author shows a more effective method of solving the problem and introduces a new case, that of thermal stresses caused by uneven distribution of the temperature throughout the parallelepiped. T. Leser.

Moisil, Gr. C. Integrals of the equations of elastic equilibrium. I. Integrals characterized by geometric conditions concerning displacements. Acad. Repub. Pop. Române. Bul. Ști. Ser. Mat. Fiz. Chim. 2, 283-291 (1950). (Romanian. Russian and French summaries)

Consider an elastic body in equilibrium, and let V denote the displacement vector (u, v, w) . The author shows that under certain circumstances, the integration of the Lamé equations for three-dimensional elasticity in the three unknown functions u, v, w may be reduced to the integration of an analogous Lamé system for two unknown functions in two independent variables (plane elasticity). This is shown, for example, when there exists a family of stream surfaces (i.e. surfaces tangent to the displacement vector V) which consists of parallel planes. Various other possibilities for stream surfaces, apparent stream surfaces, stream lines and apparent stream lines are also discussed. *J. B. Diaz.*

Berio, Angelo. Sulle equazioni di equilibrio e di congruenza delle piastre. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 329-370 (1950).

By truncating general series developments in powers of the ratio (thickness/least linear dimension), W. Z. Chien [Quart. Appl. Math. 1, 297-327 (1944); 2, 43-59 (1944); these Rev. 5, 195, 250] has classified the equations governing the deformation of plates into twelve types. The author objects to the arbitrariness of this formal procedure. He prefers to consider the phenomena directly and to give some mechanical or geometrical reason for the neglect or retention of each term. If in the equilibrium equations the second fundamental tensor of the bent plate may be neglected, he calls the deflection "infinitesimal"; if not, but if squares of the displacement gradients be negligible, he calls the deflection "small"; in all remaining cases the deflection is "finite". In the author's approach as in any other, the difficulty lies not in the equilibrium equations but in the appropriate simplification of the relations between force quantities and displacements. Here, as is customary, the author uses series developments. He concludes that there are in all seven types of plate problems, the equations for five of which he works out. Unfortunately he does not indicate the differences and similarities between his results and those of Chien and others. *C. Truesdell* (Bloomington, Ind.).

Berio, Angelo. Applicazione del teorema del minimo lavoro allo studio delle volte-membrane staticamente indeterminate. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 297-310 (1950).

For the extensional or "membrane" theory of thin shells the author calculates the strain energy in general and then shows by examples how the theorem of least work can yield quick solutions to statically indeterminate problems. The two examples concern a vault of circular cylindrical section. *C. Truesdell* (Bloomington, Ind.).

Münz, H. Ein Integrationsverfahren für die Berechnung der Biegespannungen achsensymmetrischer Schalen unter achsensymmetrischer Belastung. I. Ing.-Arch. 19, 103-117 (1951).

This paper is concerned with the integration theory of the differential equations determining small axis-symmetrical deflections of thin elastic shells of revolution. The developments are patterned along the lines of the integration theory of the canonical differential equations of analytical mechanics. The system of equations in question appears as a canonical system of six linear differential equations of first

order. The author shows that an integral of the canonical system can be found which is linear in the "impulses" and that this permits reduction of the sixth-order system to a fourth-order system. This integral is shown to be just the relation used by H. Reissner and E. Meissner in their reduction of the shell problem to a fourth-order system by way of two simultaneous second-order differential equations. The author finally formulates the shell problem in terms of four simultaneous first-order equations instead of the usual two simultaneous second-order equations. *E. Reissner.*

Münz, H. Ein Integrationsverfahren für die Berechnung der Biegespannungen achsensymmetrischer Schalen unter achsensymmetrischer Belastung. II. Ing.-Arch. 19, 255-270 (1951).

In the second part of the paper the author considers the problem of the integration of the system of four first-order equations. An iteration method and the method of asymptotic integration is discussed for the solutions of the homogeneous system. Particular integrals of the non-homogeneous equations are obtained by the method of variation of parameters. Before applying these methods, the author simplifies the differential equations somewhat, at least for shells which are not shallow, by taking account of the fact that the ratio of wall thickness to radii of curvature is small compared to unity. The physical meaning of these (known) simplifications is also indicated. [It seems to the reviewer preferable to use the method of asymptotic integration without omitting terms in advance since this leads to asymptotic solutions of somewhat better accuracy without increase in complexity.] The paper also contains numerical calculations for two specific problems of a corrugated cylindrical shell. The wall thickness is taken sufficiently large so that the method of iteration is applicable. *E. Reissner* (Cambridge, Mass.).

Aymerich, Giuseppe. Sull'espressione in coordinate curvilinee degli sforzi e degli spostamenti in elasticità piana. Rend. Sem. Fac. Sci. Univ. Cagliari 20 (1950), 175-183 (1951).

The author solves the following questions in plane elasticity: (i) he expresses the stresses in terms of Airy's stress function U in orthogonal curvilinear coordinates; (ii) he shows how to calculate the displacements from a knowledge of U and its biharmonic conjugate. He then points out how this work can be employed to obtain solutions of plane elasticity problems by conformal mapping methods, by an improvement of the techniques developed by Mushelišvili [see, for example, Math. Ann. 107, 282-312 (1932)]. Aside from the methods employed to derive the results, there is little new in this paper. *A. W. Sáenz.*

Ökubo, H. The stress distribution in an aeolotropic circular disk compressed diametrically. J. Math. Physics 31, 75-83 (1952).

The disc is taken to be of aeolotropic material which has two directions of symmetry at right angles in the plane of the plate, the fundamental equation for the stress function x in such a case being

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{k_1^2} \frac{\partial^2}{\partial y^2}\right) \left(\frac{\partial^2}{\partial x^2} + \frac{1}{k_2^2} \frac{\partial^2}{\partial y^2}\right) x = 0.$$

Expressing the stresses at any point in terms of the real parts of the two functions $\varphi_1(x+ik_1y)$, $\varphi_2(x+ik_2y)$, the author uses curvilinear coordinates defined by $x+iy = e^{\alpha+i\theta}$;

$x + ik_1y = c' \cosh(\alpha' + i\beta')$; $x + ik_2y = c'' \cosh(\alpha'' + i\beta'')$, and expresses the boundary conditions in terms of them when the disc is compressed diametrically by a pair of opposite forces P in a direction making an angle θ with an axis of symmetry. Expressing the functions φ_1 and φ_2 in the forms

$$\varphi_1(x + ik_1y) = \sum_{n=1}^{\infty} A_n \cosh 2n(\alpha' + i\beta'),$$

$$\varphi_2(x + ik_2y) = \sum_{n=1}^{\infty} B_n \cosh 2n(\alpha'' + i\beta''),$$

the coefficients A_n , B_n can be determined as infinite series from the boundary conditions. As a numerical example the stress distribution in a disc of oak is determined for the three cases $\theta = 0, \pi/2, \pi/4$. As in the isotropic case it is found that there is a similarity in the stress distribution between the circular disc and the square plate. *R. M. Morris.*

Prusakov, A. P. The fundamental equations of deflection and stability of three-layered plates with a light filler. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 15, 27-36 (1951). (Russian)

The author considers a three-layered plate with two outer thin layers made of comparatively heavy and rigid material and a thick, light and non-rigid inner layer. A. L. Rabinovič [Trudy Central. Aero-Gidrodinam. Inst., no. 595 (1946)] studied a similar plate of an infinite width; E. Reissner [J. Aeronaut. Sci. 15, 435-440 (1948); these Rev. 10, 273] solves an almost identical problem neglecting some deformations of the inner layer and assuming that the outer layers are stressed uniformly with respect to their thickness. The author of this work treats the plate as a body with orthogonal anisotropy and stepwise variation of the elasticity modulus through a cross-section of the plate. He uses the Hooke's Law modified for anisotropic bodies by S. G. Lechnitzki [Anisotropic Plates, Gostehizdat, Moscow-Leningrad, 1947; these Rev. 10, 415]. The elasticity modulus of the inner layer is very small as compared with that of the outer layers. Whenever it is convenient the author assumes that the former equals zero and the latter is infinite. Assuming further that the deflections are small the author solves the problem for three cases: (a) the edge is freely supported, (b) the edge is walled in, (c) the edge is completely free. In the last case it is not very clear what the author means by the edge being completely free. The stability in compression is also considered. *T. Leser (Lexington, Ky.).*

Nardini, Renato. Sulla linea elastica di una trave pressoinflessa in presenza di fenomeni ereditari. *Rend. Sem. Mat. Univ. Padova* 20, 286-298 (1951).

For an elastica governed by Volterra's accumulative theory of elasticity, the author proves the existence and uniqueness of solution of a class of boundary problems. The analysis rests upon a theorem of the author's on functional equations [Ann. Scuola Norm. Super. Pisa (2) 9, 201-213 (1940); these Rev. 3, 152.] *C. A. Truesdell.*

Vălcovici, V. On the elastic line of a bar in a discontinuous field. *Acad. Repub. Pop. Române. Bul. Ști. Ser. Mat. Fiz. Chim.* 2, 441-452 (1950). (Romanian. Russian and French summaries)

Vălcovici, Victor. Superposition principle for elastic bars. *Acad. Repub. Pop. Române. Bul. Ști. Ser. Mat. Fiz. Chim.* 2, 519-526 (1950). (Romanian. Russian and French summaries)

Vălcovici, V. On the buckling of elastic bars. *Acad. Repub. Pop. Române. Bul. Ști. Ser. Mat. Fiz. Chim.* 2, 167-176 (1950). (Romanian. Russian and French summaries)

Vălcovici, Victor. Energy aspect of the buckling of elastic bars. *Acad. Repub. Pop. Române. Bul. Ști. Ser. Mat. Fiz. Chim.* 2, 219-229 (1950). (Romanian. Russian and French summaries)

Lattanzi, Filippo. Applicazione della teoria dell'ellisse di elasticità trasversale allo studio di un'asta curva elasticamente vincolata agli estremi. II. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) 11, 45-52 (1951).

Cette note fait suite à une note déjà analysée [mêmes Rend. (8) 10, 395-400 (1951); ces Rev. 13, 601], dont elle conserve toute la numération. En partant des formules obtenues pour les rotations aux extrémités de la barre à cause de forces et de moments donnés, agissant séparément sur elles, au moyen d'une suite de transformations algébriques sans difficultés essentielles, on parvient à exprimer inversement les moments et la résultante des forces en fonction des déformations et des constantes caractéristiques de la barre. On considère en terminant, le cas particulier de la barre rectiligne. *B. Levi (Rosario).*

L'vin, Ya. B. On the calculation of elastic systems with two characteristics by the method of initial parameters. *Akad. Nauk SSSR. Inženernyi Sbornik* 5, no. 2, 96-102 (1949). (Russian)

The present paper is concerned with the solution of the ordinary differential equation

$$\frac{d^4 W}{d\xi^4} - 2r^2 \frac{d^2 W}{d\xi^2} + s^4 W - g = 0,$$

where r and s are constants, $W = W(\xi)$ is the unknown function, $\xi = x/l$ is a dimensionless variable, and $g = g(\xi)$ is a given function. At each of the ends $\xi = 0$ and $\xi = 1$, two of the following quantities are prescribed:

$$\varphi = \frac{1}{l} \frac{dW}{d\xi}; \quad M = -\frac{A}{P} \left(\frac{d^2 W}{d\xi^2} - \nu r^2 W \right);$$

$$Q = -\frac{A}{P} \left[\frac{d^3 W}{d\xi^3} - (2 - \nu) r^2 \frac{dW}{d\xi} \right]; \quad W;$$

where ν and A are constants, the first being Poisson's ratio. This problem includes as special cases various problems in the theory of thin beams. *J. B. Dias.*

Boženko, A. S. The bending (according to St. Venant) of a beam with a cross section composed of rectangular pieces subject to a normal force in the plane of symmetry. *Akad. Nauk SSSR. Inženernyi Sbornik* 5, no. 1, 93-107 (1948). (Russian)

The author considers a beam with a constant cross-section as in the title, one end fixed, the other end loaded by a constant force which acts in the plane of symmetry, normal to the axis of the beam. He treats the problem as a three-dimensional one and studies the following cross-sections: (1) Tee; (2) I, with different upper and lower shelves; (3) channel; (4) a cross with two axes of symmetry. In all cases the corners are sharp, hence the results cannot be used in practice. The expressions for the shearing stresses are obtained from the conventional equations of the theory of elasticity, the boundary conditions, and the conditions at

the junctions of the two rectangular regions of the cross-section. In each case the stress function contains a sum of harmonic polynomials of third degree or less plus a Fourier series. The determination of the coefficients of the Fourier series seems to be very complicated. The author compared a few numerical examples solved as two-dimensional problems with his results. The shearing stresses in the latter case are by 2% or less lower.

T. Leser.

Sonntag, G. Der Übergang vom ebenen Spannungszustand zum ebenen Formänderungszustand im breiten gebogenen Balken. *Z. Angew. Math. Mech.* 31, 344-348 (1951). (German. Russian summary)

Standard product solutions combining trigonometric and hyperbolic functions are applied to transversely loaded rectangular plates with two parallel free edges. Loads are constant in a direction perpendicular to these edges. The transition from the beam type of deformation with anti-elastic curvature to the cylindrical bending of infinitely wide plates is charted for (almost) uniform load and for constant end bending moment. Curves are drawn for the ratio of the computed anti-elastic restraint moment to the fully developed moment for various aspect ratios. As an example, at the center of a square plate, this moment is about 2/3 as large as for the infinitely wide plate.

D. C. Drucker.

Kosmodamianskii, A. S. Bending of a plane curvilinear anisotropic beam by a force applied at the end. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 16, 249-252 (1952). (Russian)

The problem of the title was first solved by Lehnickii [Anisotropic plates, Moscow-Leningrad, 1947; these Rev. 10, 415] for orthotropic beams. The author solves it for a non-orthotropic beam. The beam under consideration is a circular segment of a constant rectangular cross-section with cylindrical anisotropy. The deformations are assumed small. The author finds the expressions for stresses and gives a numerical example for chosen values of elastic constants. The solutions show the following. 1) The maximum value of the tangential stress is much greater than that of radial and shearing stresses. 2) This maximum is at a point on the inside radius and is not at the cross-section perpendicular to the direction of the bending force (the case of isotropic, orthotropic beam). 3) At the point of maximum tangential stress the shearing stress does not vanish (the case of isotropic, orthotropic beam). 4) For small ratios of the outside radius to the inside radius the distribution of stresses across a radial cross-section is as follows: tangential stresses are approximately linear, shearing stresses parabolic.

T. Leser (Lexington, Ky.).

Gorbunov-Posadov, M. I. The torsion of a beam on an elastic semi-space. *Akad. Nauk SSSR. Inzhenernyi Sbornik* 10, 187-190 (1951). (Russian)

Abramyan, B. L., and Arutyunyan, N. H. The torsion of prismatic bars with normal section in the form of a trapezoid. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 15, 97-102 (1951). (Russian)

The authors consider two cases: the normal section a rectangular trapezoid, and the normal section an isosceles trapezoid. In both cases the acute angles equal 45°. The second case, due to the symmetry can be reduced to the first one. The bar is subjected to pure torsion. The solution is obtained from the partial differential equation $\nabla^2 U = -2$, where $U(x, y)$ is the stress function, with given boundary

conditions. The rigidity and the stresses are determined when $U(x, y)$ is known. The author computes the stresses at one point only, the midpoint of the base. Numerical examples for several ratios of the cross-section dimensions are presented.

T. Leser (Lexington, Ky.).

Ševlyakov, Yu. A. On the stresses in circular rings. *Dopovidi Akad. Nauk Ukrain. RSR.* 1950, 221-224 (1950). (Ukrainian. Russian summary)

The problem of a circular ring compressed by concentrated radial forces was solved by conventional method by G. Bell [*Z. Angew. Math. Mech.* 10, 52-72 (1930); pp. 52-57] and differently by D. V. Vainberg [*Akad. Nauk SSSR. Prikl. Mat. Meh.* 13, 151-158 (1949); these Rev. 11, 68]. The author solves it using analytic functions of a complex variable, which he calls the Kolosov-Mushelishvili functions.

T. Leser (Lexington, Ky.).

Ševlyakov, Yu. A. On the stresses in the stretching of a circular ring. *Dopovidi Akad. Nauk Ukrain. RSR.* 1950, 217-220 (1950). (Ukrainian. Russian summary)

The author solves the problem of a circular ring extended by concentrated radial forces whose points of application are on the interior circle. He uses the analytic functions of a complex variable in the same way as in the paper reviewed above. A numerical example illustrates the solution.

T. Leser (Lexington, Ky.).

Alekseev, S. A. Ring-shaped elastic membrane under the action of a normal force applied to a rigid centrally located disc. *Akad. Nauk SSSR. Inzhenernyi Sbornik* 10, 71-80 (1951). (Russian)

Pestel, E. Tragwerksauslenkung unter bewegter Last. *Ing.-Arch.* 19, 378-383 (1951).

The case treated is that of a point mass and oscillatory force moving over a slightly curved beam. The mass is assumed kept in contact with the beam. Acceleration of the mass is considered but only transverse motion of beam is taken into account. Deflection is expressed as an infinite sum of products of a time function and a shape function, the latter given by the natural modes of the beam without the mass. An iteration process is proposed for each chosen position of the mass. The mass itself is replaced by a force for which a formula is given. It is stated that the first three terms of the product series are always sufficient unless the frequency of pulsating force coincides with a higher natural frequency.

D. C. Drucker (Providence, R. I.).

Ziegler, Hans. Knickung gerader Stäbe unter Torsion. *Z. Angew. Math. Physik* 3, 96-119 (1952).

The author treats stability problems of thin rods subjected to combined torsion and thrust by not requiring that the moment vectors of the external forces remain parallel to the axis. When these moment vectors are inclined, their slopes will not coincide in general with the deflection curve, since they depend on the manner in which the external forces are applied. Three types of moment vector are introduced: pseudotangential, quasitangential, and semitangential, which in contrast with the purely tangential or axial type, are such that the system is conservative and the variation of the energy of deformation is equal to the virtual work of the applied forces. The stability criterion is then that smallest load for which an equilibrium position in the deformed state exists. In this manner some new values of the critical loads are found.

D. L. Holl (Ames, Iowa).

Munakata, K. On the vibration and elastic stability of a rectangular plate clamped at its four edges. *J. Math. Physics* 31, 69-74 (1952).

The author presents a method of analyzing rectangular plate problems by use of a conformal transformation in which the boundary becomes a circle. The vibration modes of a circular plate are then used as orthogonal functions in Galerkin's method for determining deflections and eigenvalues. Numerical results obtained for three problems of a clamped square plate, using the lowest five orthogonal functions, are within 1 percent of values given previously in the literature. These problems are: (1) frequency of fundamental mode of vibration; (2) buckling load when subjected to uniform compressive forces at its four edges; and (3) deflection under uniform surface pressure. The author states numerical computations have been carried out for other cases and recommends the method for treating more complicated problems such as vibration in the presence of edge forces.

S. Levy (Washington, D. C.).

Grigolyuk, È. I. The stability of circular annular plates. *Akad. Nauk SSSR. Inženernyi Sbornik* 5, no. 2, 83-95 (1949). (Russian)

The stability of a thin circular annular plate, compressed by constant distributed loads in the plane of the plate and normal to the edges, was investigated by G. H. Bryan [*Proc. London Math. Soc.* 22, 54-67 (1891)], by A. Dannik, by A. Nádai [*Z. Verein. Deutsch. Ingenieure* 59, 169-174 (1915)], by A. Lokchine [*C. R. Acad. Sci. Paris* 189, 316-317 (1929)], and by E. Meissner [*Schweiz. Bauztg.* 101, 87-89 (1933)]. The values of critical loads are found among others from the differential equation of equilibrium, which in turn has to be solved in terms of Bessel functions. All the above investigators obtained the exact solutions by using Bessel functions. The author of this paper uses an approximate method, which he calls the method of Galerkin, and solves the problem for several different kinds of supports at the outer and inner edges. He compared his results with the exact solutions of the previous investigators, and they differed by 2 per cent to 4 per cent at the most. The author claims that the Galerkin method saves a great amount of computational work connected with the evaluation of the Bessel functions.

T. Leser (Lexington, Ky.).

Grigolyuk, È. I. Some problems of stability of circular plates with nonuniform heating. *Akad. Nauk SSSR. Inženernyi Sbornik* 6, 73-84 (1950). (Russian)

A thin annular disc of constant thickness, its center coinciding with the pole of the coordinate system, is heated in such a way that the temperature is a function of the radial distance r only. Under this condition we have a two-dimensional problem with the stress distribution symmetrical with respect to the center of the disc. The following cases are considered: (1) the outer edge freely supported; (2) the outer edge fixed, but able to expand; (3) the outer edge rigidly fixed; (4) the outer edge hinged. In all cases the temperature is given by the formula, $t = t_0 + t_1(1 - \rho)^n$, where $\rho = r/b$, and b is the outer radius of the disc. For all cases the critical temperature at the inner edge and the critical stresses are found. Solving the differential equations with given boundary conditions, the author uses a method, which he calls the method of Galerkin, but gives no reference. The author gives also tables and graphs for the stresses (in dimensionless form), and the coefficients in the formula for the critical temperature, depending on different values of n

and $\alpha = a/b$, where a is the inner radius, b is the outer radius of the disc. All the tables and graphs were calculated assuming that the Poisson ratio equals $1/3$. A numerical example for $n=4$ is appended at the end.

T. Leser.

Demidovič, B. P. The vibration of bars bent into an arc of a circle. *Akad. Nauk SSSR. Inženernyi Sbornik* 5, no. 2, 112-132 (1949). (Russian)

The vibration of circular arcs, treated as a two-dimensional problem (the vibrations in the plane of the arc) was investigated by H. Lamb [*Proc. London Math. Soc.* 19, 365-376 (1888)], I. P. Den Hartog [*Philos. Mag.* (7) 5, 400-408 (1928)], F. W. Waltking [*Ing.-Arch.* 5, 429-449 (1934)], and T. Ikebe [*Sci. Papers Inst. Phys. Chem. Res.* 34, 679-712 (1938)]. All the above mentioned investigators assumed that the bar is thin as compared with its length, thus neglecting the rotation of a cross-section, and that the central angle subtended by the arc is small. The elongation of the bar was neglected by some and taken into account by the others. The author of this paper neglects the elongation of the bar, neglects the rotation of a cross-section, but makes no assumption with regard to the central angle subtended by the arc; therefore some of his results can be applied to a circular spiral of a very small pitch. Using the conventional equations of motion including the variational principle of Hamilton, the author obtains a linear differential equation with constant coefficients of the 6th order. In the absence of external loads the right member is zero and the equation becomes homogeneous. The solution depends on the boundary conditions, which are given, and the character of the roots of the subsidiary equation. The author discusses at great length the different solutions depending on the different types of roots. The author investigates also the vibrations forced by a continuously distributed external load varying with time. This case requires finding the particular integral of the above mentioned differential equation. The main contribution is the method of solving the differential equation of motion.

T. Leser.

Castoldi, Luigi. Vibrazioni di una corda omogenea soggetta a resistenza del mezzo ambiente e a viscosità interna. *Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat.* (3) 13(82), 488-496 (1949).

The author studies small transverse motions of a taut homogeneous cord which is subject to viscous friction from the ambient medium, and is also subject to internal friction. In the case in which there are no applied transverse forces, the partial differential equation of motion is taken to be $u_{tt} = a^2 u_{xx} + 2Fu_{xt} - 2fu_t$. Here $u(x, t)$ is the instantaneous local displacement, and a, F, f are positive constants. The effect of the internal friction is assumed to be represented by the term $2Fu_{xt}$. However, no justification for this assumption is given, and the exact nature of the author's concept of internal friction is left obscure. $u(x, t)$ is calculated for a case in which the cord is subject to an arbitrary distributed transverse force depending on t , and in which the displacement is required to satisfy certain boundary and initial conditions of considerable generality. The principal result can be restated as follows. Considering displacements of the form $f(t) \sin \alpha x$, it is found that if α lies in a certain interval (α_1, α_2) , $f(t)$ is oscillatory; and if α lies outside of the interval, $f(t)$ is non-oscillatory. The author describes this as a "filter effect". It is to be noted, however, that filtration, in the usual sense, is something different, concerning rather the properties of displacements of the form $f(x) \exp(i\omega t)$.

L. A. MacColl (New York, N. Y.).

*van de Vooren, Adriaan Isak. **Theory and practice of flutter calculations for systems with many degrees of freedom.** Thesis, Technische Hogeschool te Delft, Eduard Ijdo, Leiden, 1952. vi+102 pp.

Albeit labeled "thesis," this treatise reflects the author's 10 years' experience at the National Luchtvaart Laboratorium (Netherlands) in close collaboration with J. H. Greidanus and should be of great value to the flutter analyst. The scope is indicated by the contents. 1. Introduction. 2. The equations of motion. 3. Reduction to a system with a finite number of degrees of freedom. 4. The generalized Galerkin method. 5. The integral equation and its methods of approximate solution. 6. Analysis of errors in the modified integral equation for higher modes. 7. Approximation of higher eigenvalues and eigenvectors from an iteration sequence for the fundamental. 8. Calculation of characteristic values and vectors of algebraic matrices. 9. Perturbation methods. 10. Calculation of a smaller characteristic value without the preceding evaluation of the large characteristic values. 11. Iteration of solutions obtained by means of prescribed displacement vectors. 12. Escalator method. 13. Investigation of the roots of complex polynomial equations. 14. Numerical calculation of the modes of the transposed system and comparison of different weight vectors. 15. Application of foregoing theory to a modern aeroplane. 16. Calculation of uncoupled modes. 17. Determination of stiffness terms from measured coupled frequencies and calculation of coupled vibrations in still air. 18. Investigation of flutter vibrations. 19. Variation of elasticity in control cables and of the moment of inertia of the control column. 20. Variation of air density. 21. Recapitulation. 22. List of references. 23. List of most important symbols.

Typical of the original contributions is the analysis of §4, in which it is shown that the Galerkin method (as generalized by Greidanus) leads to second order errors in the eigenvalues if the prescribed displacement vectors approximate the true eigenvectors with an error of first order and if, at the same time, the weight vectors represent the eigenvectors of the transposed (or adjoint) system also with an error of the first order. *J. W. Miles* (Los Angeles, Calif.).

Nekrasov, A. I. **A comparative analysis of flutter calculations according to the theory of unsteady and steady flows.** Akad. Nauk SSSR. Inzhenernyi Sbornik 10, 109-168 (1951). (Russian)

Biot, M. A. **The interaction of Rayleigh and Stoneley waves in the ocean bottom.** Bull. Seismol. Soc. America 42, 81-93 (1952).

Dans un système composé d'une couche liquide recouvrant un milieu élastique indéfini peuvent, comme on sait, se propager horizontalement des ondes douées de dispersion. Pour les grandes longueurs d'ondes, elles sont voisines des ondes de Rayleigh du milieu. Pour les courtes longueurs d'ondes, elles sont voisines d'ondes "de Stoneley" dont l'amplitude décroît exponentiellement de part et d'autre du fond. Ces dernières existeraient même pour un liquide incompressible et un solide sans masse. *J. Coulomb.*

Reiner, M. **An investigation into the rheological properties of bitumen. I. Maxwell-body and elastic dispersions.** Bull. Res. Council Israel 1, 5-25 (1951).

After one-dimensional discussion of various types of ideal bodies exhibiting superposition of elastic and viscous behavior, the author describes the structure of bitumen sols and gels. On the basis of this structure he correlates these

physical bodies with two different superposition models. Still employing one-dimensional methods, he works out the integrals corresponding to simple finite strain, concluding that since the results are quite different in the two cases, rheological measurements can decide between the two types of physical structure for a given specimen. There is also a qualitative discussion of simple shear. *C. Truesdell.*

Rabotnov, Yu. N. **Stress and deformation in cyclic loading.** Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 121-122 (1952). (Russian)

Experimental data indicate that the distribution of stresses in a vibrating body differs from that due to static loading in similar conditions; the concentration of stresses in the former case is lower. This paper attempts to supply the theoretical explanation. Tensor notation is used throughout. The assumptions are as follows: 1) The stresses and the deformations obey the theory of small elasto-plastic deformations; 2) the stress-strain relation is nearly linear; 3) the ratio of the scattered energy to the total elastic energy is small; 4) the area of the loop of the hysteresis is independent of the state of stress. Using these assumptions the author derives expressions for stresses and strains. These consist of two parts. The first part is in phase with the exterior forces and can be computed from the conventional elasticity formula. The second part is out of phase by 90° and looks like the expression for thermal stresses, but it is combined with the first part differently than it is in the thermal problems. The expression for scattered work is used to determine the damping properties of the material. An example of a beam bent cyclicly by exterior forces illustrates the application of the theory. *T. Leser* (Lexington, Ky.).

Anderson, Orson L. **Conditions for the derivation of the stress deviator tensor.** Amer. J. Phys. 20, 236-242 (1952).

The author states in his abstract, "The decomposition of the stress tensor into the spherical stress tensor and the stress deviator found in theories of plasticity is shown to depend upon the postulate that the stress tensor is the tensor sum of two physically independent stress constituents. The decomposition is derived by two different methods, the first method based on energy arguments, and the second method based on cause and effect arguments." The arguments presented do not convince the reviewer that a purely mathematical operation requires any justification or much less that it is subject to any "conditions". Furthermore, even the simplest experiments in plastic straining show the need for the third stress invariant dismissed so lightly by the author. Slightly more elaborate experiments require consideration of more elaborate invariants of stress and strain. The elastic energy and simple linear cause and effect relations do not seem especially relevant to anything but linear elasticity. *D. C. Drucker* (Providence, R. I.).

Freiberger, W. **A problem in dynamic plasticity: the enlargement of a circular hole in a flat sheet.** Proc. Cambridge Philos. Soc. 48, 135-148 (1952).

Following Bethe and Taylor, the sheet is divided into a plastic and an elastic part. The plastic annulus is subdivided into two regions, one (I) in which the inertia stresses are assumed to be significant enough to warrant inclusion in the equilibrium equations while the tangential stress is zero and the other (II) where the static equations are taken to apply and the sheet thickness is constant. The material is considered to be ideally plastic and governed by the maximum

shear stress criterion. Stresses in the elastic region and in region II are determined directly by the assumptions as is the ratio of the outside radius of region II to that of region I. Major effort is devoted to region I for which the characteristics are determined, shock front conditions written down and simple solutions are discussed. The case of uniform acceleration of the boundary of the hole starting with zero velocity and zero radius and the case of uniform velocity followed by constant acceleration are described in detail.

D. C. Drucker (Providence, R. I.).

Volkov, S. D. On a condition for plasticity. *Doklady Akad. Nauk SSSR* (N.S.) 76, 371-374 (1951). (Russian)

A plasticity condition is proposed which is shown to be a generalization of the Saint-Venant and Coulomb conditions. This condition is applied to cases of simple loading and to a discussion of the fatigue limits under these loadings.

H. I. Ansoff (Santa Monica, Calif.).

Sobolev, V. H., and Sokolov, L. D. On the pressure of a rigid die on a plastic medium. *Akad. Nauk SSSR. Inženernyi Sbornik* 5, no. 2, 21-24 (1949). (Russian)

Using (what appears to this reviewer) very bad engineering approximations, the authors determine the stress distribution and the change of the surface shape in a plate of finite thickness produced by indentation of a rigid stamp.

H. I. Ansoff (Santa Monica, Calif.).

Popov, S. M. On the extension of the method of relaxation of boundary conditions to the stability of a rectangular plate beyond the elastic limit. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 15, 103-106 (1951). (Russian)

A method of weakened boundary conditions is used to obtain an approximation to the critical value of rigidity of a thin plate subjected to uniaxial compression. The present application of the method essentially involves substituting for the requirement that the slope at the built-in edges be zero everywhere, a requirement that a weighted average of

the slope be zero. It is stated that this method has yielded very good approximations to the exact solution in a number of computed cases. *H. I. Ansoff* (Santa Monica, Calif.).

Korenev, B. G. On the computation of beams and plates taking account of plastic deformation. *Akad. Nauk SSSR. Inženernyi Sbornik* 5, no. 1, 58-61 (1948). (Russian)

Consider a beam on an elastic foundation subjected to two concentrated transverse forces. Further, let one or both of the forces exceed the yield limit. If the beam exhibits no hardening, the value of the moment at the plastic sections will remain constant at the yield value. The elastic solution of the problem must now be modified to satisfy this condition. This is done by adding a solution of the homogeneous beam equation which satisfies the boundary condition and produces a discontinuity in the slope of the beam at the plastic sections. The same method is applied to a brittle beam which develops a transverse crack at one of the loaded points. The method is also applied to a plate on an elastic foundation with a circular loading pattern.

H. I. Ansoff (Santa Monica, Calif.).

Pisarenko, G. S. The forced normal vibrations of built-in cantilever beams taking account of hysteresis loss. *Akad. Nauk SSSR. Inženernyi Sbornik* 5, no. 1, 108-132 (1948). (Russian)

A cantilever beam of uniform rectangular cross-section is subjected to a forced vibration by a harmonic oscillation of the support. The material of the beam is assumed to exhibit hysteresis. On the assumption of small deviation from elastic behavior, first and second order approximations are determined for the equation of motion of the beam. It is shown that, for all practical purposes, the first approximation yields sufficiently accurate results. *H. I. Ansoff*.

Berezancev, V. G. On solutions of the axially symmetric problem of the limiting equilibrium of a medium having internal friction and cohesion. *Akad. Nauk SSSR. Inženernyi Sbornik* 10, 191-198 (1951). (Russian)

MATHEMATICAL PHYSICS

✓ **Houstoun, R. A.** An introduction to mathematical physics. Blackie & Son, Ltd., London and Glasgow, 1952. x+262 pp. 25s.

A considerably revised version of an earlier text with the same title [Longmans Green, London, 1912] and intended to give a concise introduction to mathematical physics. The chapter headings are: (I) Attraction; (II) Hydrodynamics; (III) Fourier series and conduction of heat; (IV) Wave motion; (V) Electromagnetic theory; (VI) Thermodynamics; (VII) Quantum theory; (VIII) The minimal principles; relativity.

* **Kurant, R., i Gil'bert, D.** *Metody matematicheskoj fiziki. Tom pervyi.* [R. Courant and D. Hilbert, *Methods of mathematical physics. Volume one.*] 3d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 476 pp. 26.15 rubles.

Translated from *Methoden der mathematischen Physik* [2d ed., Springer, Berlin, 1931]. The references at the end of each chapter have been supplemented by references to Russian literature. The translators have also added notes (pp. 451-470) to clarify or supplement parts of the text.

Finzi, Bruno. Applicazioni fisiche del calcolo tensoriale. *Rend. Sem. Mat. Fis. Milano* 21 (1950), 106-122 (1951).

This is a descriptive article. Having briefly discussed the concept of invariance in physics, and in particular invariance under a change of reference-system, the author gives illustrations of the use of the tensor calculus in the formulation of physical laws, selecting for this purpose the mechanics of continuous media, the general-relativity theories of gravitation and electromagnetism, and the recent unified field-theory of Einstein. *H. S. Ruse* (Leeds).

Graffi, Dario. Il metodo ereditario per lo studio di alcuni fenomeni fisici. *Univ. e Politecnico Torino. Rend. Sem. Mat.* 10, 51-66 (1951).

Expository lecture on the methods and results of accumulative physics, in which a field quantity is regarded as composed in part of a weighted time integral of its values after a certain epoch in the past. The emphasis is placed on dielectric hysteresis, where a generalized Hopkinson formula

$$D(t) = \epsilon E(t) + \int_0^t \phi(t, \tau) E(\tau) d\tau$$

relates the displacement D with the intensity E . Results of Volterra and the author are briefly summarized.

C. Truesdell (Bloomington, Ind.).

Optics, Electromagnetic Theory

Bremmer, H. On the theory of optical images affected by artificial influences in the focal plane. *Physica* 17, 63-70 (1951).

Viene studiato il caso di una lamina che altera la distribuzione dell'ampiezza e della fase nel piano focale dello strumento, essendo l'oggetto illuminato da una sorgente puntiforme all'infinito. Come è noto, l'ampiezza complessa nel piano immagine è data dall'integrale quadruplo

$$u_1(P) = \frac{k_0^2 e^{i k_0 \overline{P'P}}}{4\pi^2 f N^2} \iint dO_Q u(Q) \iint dO_K \varphi(K) \times \exp \left[i k_0 / f \left\{ \left(\frac{x_P}{N} - x_Q \right) x_K + \left(\frac{y_P}{N} - y_Q \right) y_K \right\} \right],$$

essendo $k_0 = 2\pi/\lambda$, f la distanza focale dello strumento, N l'ingrandimento parassiale, $u(Q)$ l'ampiezza complessa sull'oggetto e $\varphi(K)$ il fattore di trasmissione della lamina nel piano focale. L'autore esamina vari casi particolari per $\varphi(K)$. È possibile fra l'altro riprodurre nell'immagine il modulo del gradiente dell'ampiezza o della fase sull'oggetto.

G. Toraldo di Francia (Firenze).

Seman, O. I. The reduced form of the eikonal of the fourth order and of the aberration coefficients in electron optics. *Doklady Akad. Nauk SSSR (N.S.)* 81, 775-778 (1951). (Russian)

Nel caso di un campo elettrico e di un campo magnetico statico l'autore dà delle espressioni per la funzione iconale del quarto ordine e per i coefficienti delle aberrazioni, che afferma essere molto adatte per il calcolo numerico.

G. Toraldo di Francia (Firenze).

Glaser, Walter, und Robl, Hermann. Strenge Berechnung typischer elektrostatischer Elektronenlinsen. *Z. Angew. Math. Physik* 2, 444-469 (1951).

Vengono studiate le soluzioni rigorose per le traiettorie parassiali nelle lenti elettrostatiche con i potenziali dati rispettivamente da

$$\Phi(z) = U_A + \frac{U_L}{\pi} \arccot \left(-\frac{z}{a} \right),$$

$$\Phi(z) = U_A + \frac{U_L}{2} \left[1 + \frac{z/a}{1 + |z/a|} \right].$$

Vengono poi calcolate l'aberrazione sferica e l'aberrazione cromatica. Gli autori affermano che queste lenti sono molto simili a quelle cilindriche.

G. Toraldo di Francia.

Mertens, Robert. Diffraction of light by standing super-sonic waves. General theory. *Simon Stevin* 28, 164-180 (1951).

This paper is closely related to its antecedents [Simon Stevin 27, 212-230 (1950); 28, 1-12 (1951); these Rev. 13, 94], albeit without reference thereto.

J. W. Miles.

Avazašvili, D. Z. The spatial problem of diffraction of monochromatic electromagnetic waves. *Doklady Akad. Nauk SSSR (N.S.)* 82, 29-32 (1952). (Russian)

L'autore chiama così il problema delle onde elettromagnetiche in un mezzo omogeneo indefinito, nel quale sia immerso un corpo omogeneo, limitato da una superficie chiusa, essendo una parte del campo assegnata e una parte soggetta alla condizione di radiazione. Come è noto, questo problema dipende da un sistema di equazioni integrali. L'autore scrive le equazioni integrali alle quali soddisfano il potenziale vettore e il potenziale scalare rispettivamente e stabilisce l'esistenza della soluzione (che è unica).

G. Toraldo di Francia (Firenze).

Hönl, H. Eine strenge Formulierung des klassischen Beugungsproblems. *Z. Physik* 131, 290-304 (1952).

Scalar diffraction problems for plane apertures and screens (perfectly rigid or soft) are formulated, by use of Fourier transform, in terms of a pair of dual integral equations. The author gives a clear exposition but was anticipated by Clemmow [Proc. Roy. Soc. London. Ser. A. 205, 286-308 (1951); these Rev. 12, 884], for example. The analogous formulation of electromagnetic diffraction problems was given by Silver in a paper presented to the IXth General Assembly of URSI (Zürich, 1950, doc. A.G., nr. 26, comm. VI) to be published in the Proceedings of URSI.

C. J. Bouwkamp (Eindhoven).

Boivin, Albéric. On the theory of diffraction by concentric arrays of ring-shaped apertures. *J. Opt. Soc. Amer.* 42, 60-64 (1952).

La sostanza matematica di questo lavoro è costituita dalla ricerca di un metodo rapido di calcolo numerico per l'integrale

$$\Phi(a, k, l) = \int_0^a \exp(-ik\rho^2/2) \rho J_0(l\rho) d\rho$$

che di solito si valuta con le classiche funzioni di Lommel. L'autore deduce e consiglia i seguenti sviluppi in serie

$$(1) \quad \Phi(a, k, l) = \frac{1}{ik} \left\{ e^{i\pi/4} - \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{i l^2}{2k} \right)^r F_r(i k a^2/2) \right\}$$

con $F_r(x) = e^{-x} \sum_{p=0}^{\infty} x^p / p!$ e

$$(2) \quad \Phi(a, k, l) = \frac{1}{l^2} \sum_{r=0}^{\infty} (-1)^r r! \left(\frac{2ik}{l^2} \right)^r \times \sum_{p=0}^r (-1)^p \frac{(la)^{2p}}{(2^p p!)^2} \{ la J_1(la) + 2p J_0(la) \}.$$

La (1) converge rapidamente per $l^2 \leq k$ e la (2) per $l^2 \geq k$. Inoltre l'autore enuncia il seguente teorema di moltiplicazione per le funzioni di Lommel

$$V_n(\alpha^2 y, \alpha z) = \sum_{m=0}^{\infty} (-1)^m \frac{(\alpha^2 - 1)^m y^m}{2^m m!} V_{n+m}(y, z),$$

$$U_n(\alpha^2 y, \alpha z) = \sum_{m=0}^{\infty} \frac{(\alpha^2 - 1)^m y^m}{2^m m!} U_{n-m}(y, z).$$

Le formule vengono applicate al problema fisico contemplato nel titolo e, in particolare, ai reticoli zonali.

G. Toraldo di Francia (Firenze).

Twersky, Victor. Multiple scattering of radiation by an arbitrary planar configuration of parallel cylinders and by two parallel cylinders. *J. Appl. Phys.* 23, 407-414 (1952).

"The mathematical details of this paper and further discussion will be found in Multiple Scattering of Radiation, Part I" [New York University, Washington Square College, Mathematics Research Group, Research Rep. No. EM-34 (1951); these Rev. 13, 188].

Weyl, Hermann. Radiation capacity. *Proc. Nat. Acad. Sci. U. S. A.* 37, 832-836 (1951).

The author deals with the existence theorem of the first external boundary-value problem for the scalar wave equation. The method of approach is suggested by well-known facts for the potential equation. The analogue for electromagnetic problems is promised for a subsequent paper.

C. J. Bouwkamp (Eindhoven).

Eckart, Gottfried. Über die Reflexion ebener elektromagnetischer Wellen in schwach inhomogenen Schichten. *Arch. Elektr. Übertragung* 5, 555-560 (1951).

By a method credited to Schelkunoff [Quart. Appl. Math. 3, 348-355 (1946); these Rev. 7, 300] the author investigates theoretically the reflexion of electromagnetic waves by an inhomogeneous layer of slightly varying dielectric permeability. The question is discussed in how far the form of reflected pulses can give useful information about the structure of the layer. References.

C. J. Bouwkamp.

Stevenson, A. F. General theory of electromagnetic horns. *J. Appl. Phys.* 22, 1447-1460 (1951).

This paper deals with perfectly conducting horns of arbitrary shape. The exact equations are derived from Maxwell's theory and lead to wave equations in three dimensions for the electric and the magnetic field strength. Four auxiliary functions are introduced relating to the cross-section of the horns. In all, 6 functions are to be determined: the components of the electric and of the magnetic field strengths in the direction of propagation and the said four auxiliary functions. All these functions are then developed in infinite series in which each term is a product of a function of z (direction of propagation) and a function related to the cross-section. The boundary conditions then lead to equations for the various constants. In the first place E and H waves in horns of small flare, when coupling is neglected, are discussed. Then the coupling of modes is discussed. The field components are then approximately determined. The general equations are applied to horns of rectangular sections, of circular sections, of sectorial shape, of pyramidal shape, of rectangular exponential shape, of conical shape and of circular exponential shape. Some proofs of formulas used in the paper are given in an appendix.

M. J. O. Strutt (Zurich).

Zelkin, E. G. Waves in a pyramidal megaphone. *Akad. Nauk SSSR. Zhurnal Tehn. Fiz.* 21, 1228-1239 (1951). (Russian)

La piramide dell'altoparlante viene sostituita in via di approssimazione dalla figura compresa fra due semiconi della stessa apertura opposti al vertice e due semipiani uscenti dall'asse dei coni. Il problema elettromagnetico può così essere risolto in coordinate sferiche. La determinazione del numero azimutale l , corrispondente a ciascun modo, dipende da una equazione trascendente, non risolvibile in

termini finiti. L'autore assegna delle limitazioni per l , che ne consentono il calcolo approssimato. Risulta l'impossibilità di onde E_{n0} e E_{0n} . G. Toraldo di Francia (Firenze).

*Schelkunoff, Sergei A. Advanced antenna theory. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1952. xii+216 pp. \$6.50.

The author gives a unified account of recent developments in antenna theory. His principle object is to present advanced mathematical methods for solving antenna problems, while stressing the resemblance of antennas to conventional circuits and transmission lines. The greater part of the book is devoted to, or written in the spirit of, the mode theory of antennas as developed by the author himself during the last twenty years. Much hitherto unpublished work is included. An account of Stratton and Chu's theory of spheroidal antennas is presented. There is also a compact exposition of Hallén's method of obtaining asymptotic solutions for linear antennas. Chapter headings are: Spherical waves, Mode theory of antennas, Spheroidal antennas, Integral equations, Cylindrical antennas, and Natural oscillations. The book concludes with a number of problems, useful numerical tables of antenna impedance functions, and a general index.

C. J. Bouwkamp (Eindhoven).

Ahiezer, A. I., and Fainberg, Ya. B. On high frequency oscillations of an electron plasma. *Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz.* 21, 1262-1269 (1951). (Russian)

Viene studiata l'interazione di un fascio omogeneo e indefinito di particelle cariche con un plasma elettronico. Si trova che il moto del fascio di particelle è instabile, in quanto, se nascono in esso piccolissime fluttuazioni rispetto al valore medio della densità e della velocità, queste si propagano sotto forma di onde di ampiezza crescente col tempo. Lo stesso fenomeno si verifica nel passaggio del fascio di particelle attraverso un dielettrico, quando la velocità delle particelle è superiore a quella della luce nel dielettrico (cioè quando si verifica l'effetto Čerenkov).

G. Toraldo di Francia (Firenze).

Ginzburg, V. L. On magneto-hydrodynamic waves in a gas. *Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz.* 21, 788-794 (1951). (Russian)

The author examines the problem of magneto-hydrodynamic plane waves propagating through ionized gas in electromagnetic field of constant gravity. Solutions of linearized equations are obtained which, at the high-frequency end of the spectrum, correspond to the propagation of radio waves in the ionosphere under the influence of the terrestrial magnetic field, while at low frequencies they pass into a form identical with ordinary hydrodynamical approximations. The same problem has been treated with very similar results and published almost simultaneously by Åström [Ark. Fysik 2, 443-457 (1951); these Rev. 12, 778].

Z. Kopal (Manchester).

Karp, Samuel N. The natural charge distribution and capacitance of a finite conical shell. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Report No. EM-35, i+ii+52 pp. (1951).

The author's abstract is as follows: "The natural charge distribution for a conical cup has been obtained without approximation. Using spherical coordinates, the potential ψ

is expressed in the form

$$u = \int_{-\infty}^{+\infty} r A(\nu) P_{\nu-1}(\cos \theta) d\nu \quad \text{for } 0 < \theta < \theta_0,$$

and

$$u = \int_{-\infty}^{+\infty} r B(\nu) P_{\nu-1}(-\cos \theta) d\nu \quad \text{for } \theta_0 < \theta < \pi.$$

The two-part boundary value problem in r at $\theta = \theta_0$ then gives rise to a single equation between two functions of ν . This equation is solved by Wiener-Hopf techniques in the complex $\mu (= \nu - \frac{1}{2})$ plane. This involves: (a) factorization of the product $2P_{\nu-1}(\cos \theta_0)P_{\nu-1}(-\cos \theta_0)/\cos \pi\mu$ in the form $K^+(\mu)K^-(\mu)$, where $K^+(\mu)$ is regular in a right half-plane, and $K^-(\mu)$ in a left half-plane; (b) the asymptotic forms of $K^+(\mu)$ and $K^-(\mu)$. The necessary results are obtained by employing: (a) the fact that the zeros of $P_{\nu-1}(\cos \theta_0)$ considered as a function of μ are real, symmetric about the origin and asymptotically in arithmetical progression; (b) comparison of $K^+(\mu)$ with the gamma function of a suitable argument. From the knowledge of the natural charge distribution the capacitance of the conical cup is obtained as well as the behaviour of the charge densities at the apex and the circular edge of the cup."

The theory is the natural extension of that given in an earlier report by the same author for the problem of the circular disc [Research Rep. No. EM-25 in the same series; these Rev. 12, 775].
E. T. Copson (St. Andrews).

Jouvet, Bernard. Sur la théorie classique du point chargé. C. R. Acad. Sci. Paris 234, 712-714 (1952).

The author reviews some known properties of the Maxwell field equations and the Lorentz ponderomotive force equations. He proposes to use an eight-dimensional space in which the coordinates are the four space-time coordinates and the four components of the vector potential to discuss some of the invariance properties of the equations listed above.
A. H. Taub (Urbana, Ill.).

Jouvet, Bernard. Les fondements d'un nouvel électromagnétisme. C. R. Acad. Sci. Paris 234, 819-822 (1952).

The author uses an eight-dimensional space discussed in the paper reviewed above to formulate a law of superposition of electro-magnetic fields satisfying the Born-Infeld field equations and which are such that $E^2 - H^2$ is bounded by a constant b^2 if the two fields (E_1, H_1) and (E_2, H_2) of which (E, H) is the composite each satisfy this condition.
A. H. Taub (Urbana, Ill.).

Bechert, Karl. Zur nicht-linearen Elektrodynamik. Naturwissenschaften 39, 185 (1952).

Chaney, Jesse Gerald. A critical study of the circuit concept. J. Appl. Phys. 22, 1429-1436 (1951).

By applying the complex Poynting vector to a simple electrical circuit, the internal and external impedances of the circuit are derived such that the current is not required to have everywhere the same time phase; however, it is desirable to have explicitly given the current distribution. It is demonstrated that more conventional methods for evaluating antenna impedances can be derived as special cases.
E. Weber (Brooklyn, N. Y.).

Staehler, Robert E. An application of Boolean algebra to switching circuit design. Bell System Tech. J. 31, 280-305 (1952).

This paper discusses the application of switching (Boolean) algebra to the development of an all-relay dial pulse count-

ing and translating circuit employing the minimum number of relays. An attempt is made to outline what appears to be the most promising method of obtaining beneficial results from the use of the algebra in the design of practical switching circuits. (From the author's summary.)

S. Sherman (Sherman Oaks, Calif.).

Shen, D. W. C. Operational impedance matrices of n -phase partially-symmetrical machines. Australian J. Sci. Research. Ser. A, 4, 544-559 (1951).

The theory of n -phase rotating machines is analyzed by partitioning the set of n -dimensional hyper-spaces associated with a machine into two sets of orthogonal subspaces with $n-2$ and 2 dimensions respectively. The partitioning is dictated by the presence in n -phase rotating machines of a set of Riemannian and non-Riemannian spaces, each space extending, however, in two dimensions only. This set of 2-dimensional spaces is interlinked magnetically and also is interconnected by means of rotating and stationary points of contact. The remaining set of $n-2$ dimensional spaces is actually a sub-space of a set of externally connected stationary networks proper. G. Kron (Schenectady, N. Y.).

Bolinder, Folke. Fourier transforms in the theory of inhomogeneous transmission lines. Acta Polytech., no. 88=Trans. Roy. Inst. Tech. Stockholm 1951, no. 48, 84 pp. (1951).

The properties of a transmission line are determined by an ordinary differential equation of the second order. If the spacing between the wires is variable the coefficient $p(x)$ of the first derivative term is, in general, not constant. A simple approximate solution of the equation is found for a range of conditions met in practice. The main formula gives the frequency variation of the reflection coefficient of the line as a Fourier transform of $p(x)$ over the length of the line. For cases in which the equation can be solved exactly this formula checks quite well.
R. J. Duffin.

Zadeh, Lotfi A., and Miller, Kenneth S. Generalized ideal filters. J. Appl. Phys. 23, 223-228 (1952).

A projection operator which projects a function of the time (the signal) into the manifold of functions having a limited Fourier spectrum may be termed an ideal filter. The authors propose to generalize this definition so as to include arbitrary projection operators on arbitrary nonlinear manifolds. They indicate how such ideal filters could be approximately realized.
R. J. Duffin (Pittsburgh, Pa.).

Costas, John P. Coding with linear systems. Research Laboratory of electronics, Massachusetts Institute of Technology, Tech. Rep. no. 226, ii+9 pp. (1952).

This report investigates the advantage of using "pre-distortion" in a linear communication system with a noisy channel. A message signal $f_m(t)$ with power spectrum $\phi_m(\omega)$ is passed through a (pre-distorting) filter with transfer function $H(\omega)$ and then to the input of a channel containing noise with power spectrum $\phi_n(\omega)$. At the output end of the channel there is a filter with transfer function $G(\omega)$. Let $f_o(t)$ denote the signal which emerges from this filter. The filter functions $H(\omega)$ and $G(\omega)$ are to be chosen so as to minimize the mean square error

$$\lim_{T \rightarrow \infty} (2T)^{-1} \int_{-T}^T [f_o(t) - f_m(t-a)]^2 dt$$

with the side condition that the power of the transmitted

signal

$$(1) \quad P = \int_{-\infty}^{\infty} |H(\omega)|^2 \phi_m(\omega) d\omega$$

is held fixed. A solution is given for the case in which the delay a is allowed to be arbitrarily large and the cross-correlation between the message $f_m(t)$ and the channel noise is zero. The optimal pre-distortion $H(\omega)$ then satisfies

$$|H(\omega)|^2 \phi_m(\omega) = \begin{cases} (K \phi_m \phi_m)^{1/2} - \phi_m & \text{if this is } \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In the above expression, K is a constant chosen to make (1) hold. *E. N. Gilbert* (Murray Hill, N. J.).

Quantum Mechanics

*Guinier, Georges. *Éléments de physique moderne théorique*. Bordas, Paris. Tome I, pp. 1-160 (1949); tome II, pp. 161-309 (1950).

This is a textbook providing a general introduction to the main ideas of wave mechanics and of atomic and nuclear structure. Only a few of the simpler problems are treated completely, and much of the discussion is conducted in general and suggestive terms, as is to be expected in a relatively short book for beginners in the subject. The first volume begins with a description of the properties and general behavior of elementary particles and photons, with illustrations of the wave-particle dualism. It continues with a treatment of some of the problems of the non-relativistic wave mechanics of a single particle, and concludes with a discussion of the general principles of quantum mechanics. The second volume contains a description of some of the main facts of atomic structure and valence; a discussion of spectroscopy including molecular and X-ray spectra and the Zeeman and Raman effects (but not the Stark effect); and an introduction to a few elementary ideas about nuclear structure. The fact that it has neither a table of contents nor an index will lessen the usefulness of the book.

W. H. Furry (Cambridge, Mass.).

Lévy, Maurice. *Wave equations in momentum space*. Proc. Roy. Soc. London. Ser. A. **204**, 145-169 (1950).

The treatment of the Schrödinger and Dirac equations in the momentum representation is systematized by maximum use of the general integral equation of spherical and hyperspherical harmonics [E. Hecke, Math. Ann. **78**, 398-404 (1918)]. This is first applied to the separation of the angular coordinates in the central-field case. Fock's transformation [V. Fock, Z. Physik **98**, 145-154 (1935)] is then used to express the radial equations in terms of angular coordinates in a four-dimensional space, and a further application of Hecke's theorem provides the formalism for the solution in series of hyperspherical harmonics. Applications are given for the Coulomb field, the Yukawa potential, and a general potential whose representation in momentum space takes the form of power series. *W. H. Furry*.

Novobatzky, K. F. *Das klassische Modell der Quantentheorie*. Ann. Physik. (6) **9**, 406-412 (1951).

In a Lagrangian describing an ensemble of classical particles the author takes as independent variables the principal function and the square root A of the particle density, and by adding an A -dependent term shows that the equations of motion are the continuity equation and the Schrödinger

equation. Motion according to the latter then corresponds to a classical motion in an effective potential

$$V^* = V - [A^2/2m][\nabla^2 A/A]$$

[cf. Bohm, Physical Rev. (2) **85**, 166-179, 180-193 (1952); these Rev. **13**, 709, 710]. *H. C. Corben* (Genoa).

Madhava Rao, B. S. *Commutation rules for matrices related to particles of higher spin. III. Half-Yearly J. Mysore Univ. Sect. B., N.S. 6*, 57-62 (1945).

The commutation rules for particles of arbitrary spin contain a constant k , which determines the possible values of the rest mass of the particle. For the spin values $3/2$, 2 , $1/2$, 1 , the possible values of k have been determined by special methods. In this paper the author shows that for higher spins, k may be left undetermined during the derivation of the commutation rules, and while deriving the higher order wave equation satisfied by the wave function, from the first order wave equation and the commutation rules. Or k may be set arbitrarily equal to 1 for integral spin values, and equal to $1/4$ for half-integral values. The commutation rules and higher order wave equations are derived, but they are quite complicated in the general case. *O. Frink*.

Yang, L. M. *A note on the quantum rule of the harmonic oscillator*. Physical Rev. (2) **84**, 788-790 (1951).

The author investigates the extent to which the commutation rules of quantum mechanics are arbitrary, and to what extent they are determined by the other assumptions. For the simplest case of the harmonic oscillator, he shows that the usual commutation rule follows from the assumption that the wave function representing a physically admissible state is always expandable in a uniformly and absolutely convergent series of eigenfunctions satisfying the natural boundary condition (vanishing at infinity). However, if it is assumed merely that this series must converge in the mean, then other (presumably incorrect) commutation rules seem to be allowable. *O. Frink* (State College, Pa.).

Gupta, Suraj N. *Quantization of Einstein's gravitational field: linear approximation*. Proc. Phys. Soc. Sect. A. **65**, 161-169 (1952).

The field equations of general relativity are first linearized and then quantized with the aid of an indefinite metric in Hilbert space. The supplementary conditions which are used to rule out the occurrence of observable states with negative probabilities are taken to be the usual coordinate system conditions used in the classical linearized theory. The observable gravitons are shown to be particles of spin 2. The interaction of the gravitational field with the matter field is briefly discussed by means of the interaction representation.

A. H. Taub (Urbana, Ill.).

Demeur, M. *Solutions singulières des équations de Klein-Gordon et de Dirac, tenant compte d'un champ électrique extérieur*. Physica **17**, 933-937 (1951).

Exact singular solutions of the wave equations satisfied by charged particles of spin 0 and $\frac{1}{2}$ in a uniform and constant electric field are obtained. *A. H. Taub*.

Petiau, Gérard. *Sur la résolution des équations d'ondes du corpuscule de spin $\frac{1}{2}$ en interaction avec un potentiel pseudoscalaire radial*. J. Phys. Radium (8) **12**, 810-816 (1951).

Solutions are derived for the Dirac equation with an interaction term $U(r) + I_1(r)\alpha_4 + I_2(r)\alpha_5$, r being radial distance,

i.e. the interaction contains an electrostatic potential, a scalar potential and a pseudoscalar potential. The equation has the usual angular momentum integral. Discussion is confined to the cases of all potentials constant, U, I_1 constant and $I_2 = \text{const.} \times r^{-1}$, all potentials proportional to r^{-1} , U, I_1 constant and I_2 an arbitrary function of r .

C. Strachan (Aberdeen.)

Petiau, Gérard. Sur la représentation des équations d'ondes des corpuscules de spin 0 ou $\frac{1}{2}$. C. R. Acad. Sci. Paris 234, 1534-1537 (1952).

Le Couteur, K. J. Factorization of the algebra of particles of half-odd spin. Proc. Cambridge Philos. Soc. 48, 110-117 (1952).

The author proves that the matrix algebra for any relativistic wave equation of half-odd integer spin can be factored into the direct product of a Dirac algebra and another which he calls the ξ algebra. This factorization cannot be done for integral spins. The ξ algebra contains only one sixteenth as many elements as the original matrix algebra.

A. H. Taub (Urbana, Ill.).

Wessel, Walter. Zur relativistischen Quantenmechanik. II. Z. Naturforschung 6a, 473-477 (1951).

In a previous paper [same Z. 4a, 645-653 (1949); these Rev. 11, 577] the author introduced a mass operator which he defined as a function of two invariants of a representation of the Lorentz group. This paper is devoted to a discussion of some general properties of this operator and the evaluation of it and some quantities related to it in various infinite-dimensional unitary representations of the Lorentz group.

A. H. Taub (Urbana, Ill.).

Blohinev, D. On the propagation of signals in nonlinear field theory. Doklady Akad. Nauk SSSR (N.S.) 82, 553-556 (1952). (Russian)

The author considers the classical theory of a scalar field ψ based on a generalized Lagrangian

$$L = L(K, I), \quad K = \frac{1}{2}[(\partial\psi/\partial t)^2 - (\nabla\psi)^2], \quad I = \frac{1}{2}\psi^2.$$

In a linear theory L will be a linear function of K , and the field-equation will be a hyperbolic equation with characteristic surfaces everywhere coinciding with light-cones. In this case all signals will be transmitted with velocities not exceeding the light-velocity. But suppose L is not a linear function of K . Let $\xi = (dx/dt)$ be the direction of a characteristic surface describing propagation in the x -direction, i.e. independent of the other two coordinates y and z . Suppose $\alpha = (\partial^2 L / \partial K^2) / (\partial L / \partial K)$ is chosen to be small compared to unity. Then

$$\xi = \pm [1 - \frac{1}{2}\alpha((\partial\psi/\partial t) \pm (\partial\psi/\partial x))^2]$$

gives the two possible values of ξ . When α is negative we have in general $|\xi| > 1$, meaning that signals will be propagated with greater-than-light velocity.

An analysis of non-linear electrodynamics, a similar generalization of the Maxwell theory, shows that greater-than-light velocities will occur in exactly the same way there too. The author concludes by saying that it is hopeless to attempt to construct non-linear versions of quantum electrodynamics until these simple classical effects of the non-linearities are properly understood.

F. J. Dyson.

Valatin, J. G. On quantum electrodynamics. Danske Vid. Selsk. Mat.-Fys. Medd. 26, no. 13, 32 pp. (1951).

A relativistically covariant formulation of quantum electrodynamics is given in the interaction representation by using the six-vector solutions of the vacuum Maxwell equations to characterize transverse photon states and by introducing another scalar field to characterize longitudinal photons. If the variables of the latter field are eliminated by a canonical transformation the usual formulation results. In the Heisenberg representation the potentials do not satisfy the Lorentz condition but the field strengths are gauge independent and independent of the scalar field variables. It is not necessary at any stage to impose a supplementary condition.

H. C. Corben (Genoa).

Dyson, F. J. Divergence of perturbation theory in quantum electrodynamics. Physical Rev. (2) 85, 631-632 (1952).

An argument is advanced which "suggests" that the series expansions of current quantum electrodynamics (qed.) diverge. The author has previously [same Rev. 83, 608-627, 1207-1216 (1951); these Rev. 13, 608, 609] proposed a method of carrying out the renormalization program which expresses any electrodynamic quantity F as a series in even powers of the elementary electric charge e ,

$$F = a_0 + a_2 e^2 + a_4 e^4 + \dots,$$

with finite coefficients a_n . In §XII of the first of the papers referred to above the author hazarded the opinion that this series would converge. He now argues that if it did so $F(e^2)$, regarded as a complex function of e^2 , would be regular at $e^2 = 0$ and hence the series would converge for e^2 replaced by $-e^2$. This could occur in a universe in which like charges attracted each other. By a physical argument, based on the tunnel effect, Dyson concludes that the field quantities of this fictitious world would not be analytic functions of e^2 . Thus the series for $F(-e^2)$ and therefore for $F(e^2)$ cannot converge in any neighbourhood of $e^2 = 0$. Assuming this conclusion is correct, the author sees two possibilities, (A) current qed. is essentially complete but $F(e^2)$ must be expressed in some other form for which the divergent power series is an asymptotic expression, or (B) current qed. is incomplete and a new physical theory is needed. Alternative B is more consonant with our experience that whereas there is a large group of phenomena to which qed. apparently applies with great accuracy, there is another, not unrelated to the first, which we do not understand at all. This suggests that current qed. is incomplete.

A. J. Coleman (Toronto, Ont.).

Fabre de la Ripelle, Michel. Résolution des équations de perturbation. I. Les amplitudes. C. R. Acad. Sci. Paris 234, 412-414 (1952).

By writing the coefficients of the unperturbed solutions in the expansion of the perturbed solutions in the form

$$a_m(t) = \int_0^t g_m(\tau) e^{-i\nu_m(t-\tau)} d\tau$$

with ν_m the sum of the average value of the total Hamiltonian in the state m and another term $\nu_m^{(0)}$ it is shown how to obtain recurrence formulae for the Fourier transforms of the g_m .

H. C. Corben (Genoa).

Iwata, Giiti. Relativity of representation coordinates and its consequences. *Progress Theoret. Physics* 6, 684-690 (1951).

Gauge transformations of the second kind are extended to transformations of the type

$$\psi(x) \rightarrow \psi'(x) = \int (x|u|x')\psi(x')dx'$$

(u unitary) under which $\int \bar{\psi}\psi dx$ is invariant. The Dirac Lagrangian is invariant under this generalized transformation if the charge is represented by a skew-symmetric hermitean matrix. Some consequences of this in quantum electrodynamics are discussed. *H. C. Corben (Genoa).*

Albertoni, S. Osservazioni sull'equazione fondamentale dell'elettrodinamica di Tomonaga-Schwinger. *Nuovo Cimento, Supplement* (9) 8, 168-179 (1951).

An expository article, describing the method of Tomonaga [*Progress Theoret. Physics* 1, 27-42 (1946); these *Rev.* 10, 226] for setting up the equations of motion of quantum electrodynamics in a manifestly covariant form.

F. J. Dyson (Ithaca, N. Y.).

Chisholm, J. S. R. Calculation of S -matrix elements. *Proc. Cambridge Philos. Soc.* 48, 300-315 (1952).

A new method is proposed for carrying out the evaluation of matrix elements for processes of high order in quantum electrodynamics or in any quantized field theory. The method is to reduce all such matrix elements to a standard form in which certain differential and integral operators operate on an integrand which is merely a negative integral power of a certain quadratic form in the momentum variables. By diagonalizing the quadratic form the integration over the momentum variables can be performed easily. However, there then remain numerous integrations to be carried out with respect to various auxiliary parameters, and these subsequent integrations usually constitute the major part of any large-scale calculation. Hence it is doubtful whether the author's method will, as he hopes, materially reduce the amount of labor involved in practical calculations. In the reviewer's opinion the chief advantage of the method is that it makes possible a more precise and systematic estimation of the effects of high-order matrix elements when these are too numerous or complicated to be calculated explicitly. Also it is possible that the new method might enable progress to be made with the problem of coding field-theory calculations for automatic handling by mechanical computers. *F. J. Dyson (Ithaca, N. Y.).*

de Broglie, Louis. Sur le tenseur énergie-impulsion dans la théorie du champ soustractif. *C. R. Acad. Sci. Paris* 234, 20-22 (1952).

A new form is proposed for the energy-momentum tensor $T^{\mu\nu}$ of the author's subtractive field theory [same *C. R.* 232, 1269-1272 (1951); these *Rev.* 12, 890]. The new tensor has the advantage that the divergence $T^{\mu\nu}_{;\nu}$ is now zero even inside a charged source, and not only in the space exterior to the sources. All other consequences of the theory are unchanged. *F. J. Dyson (Ithaca, N. Y.).*

Araki, Gentaro, and Huzinaga, Sigeru. Recoil effect on electron-proton forces and inapplicability of energy law. *Progress Theoret. Physics* 6, 673-683 (1951).

The authors clear up several long-standing confusions in the theory of the electromagnetic interaction between two

charged particles. First they give a satisfactory and unique definition of what is meant by the correct interaction, as follows. Starting from the usual formalism of quantum electrodynamics, we have a total Hamiltonian H which is partly diagonal and partly non-diagonal with respect to the number of photons present. We choose a totally non-diagonal and skew-Hermitian operator S such that the operator

$$(\exp S)H(\exp(-S)) = H_0 + W$$

is diagonal in the photon-numbers, H_0 being the Hamiltonian of the particles without interaction. These conditions fix S and W uniquely, and we can calculate W by successive approximation in powers of the elementary charge e . By definition W is the particle interaction operator.

Two approximate interactions have been widely used in the past, that of Breit [*Physical Rev.* (2) 34, 553-573 (1929)] and that of Møller [*Z. Physik* 70, 786-795 (1931)]. It has never been clear how far the approximations made in deducing these interactions are justified in the circumstances in which they have been used. The Breit formula involves neglect of particle recoil, the Møller formula assumes exact energy conservation during the interaction, neither assumption being generally valid. The authors here show precisely what are the differences between the Breit and Møller formulae and the correct interaction W calculated to the order e^4 . The differences are easily large enough to be observable, the difference between W and the Breit interaction being apparent in the hydrogen hyperfine structure, and that between W and the Møller interaction appearing in the hydrogen fine-structure. In both cases the experiments confirm the correctness of W . *F. J. Dyson.*

Sachs, R. G., and Dexter, D. L. Quantum limits of the electrostatic image force theory. *J. Appl. Phys.* 21, 1304-1308 (1950).

The force between a small charged body and a metal surface is given in traditional classical theory by the method of images. Because of the ionic-electronic structure of the metal, this result becomes inaccurate when the ratio of the distance to atomic dimensions is not large compared with the square root of the ratio of the charge to the charge of an electron. Deviations are caused by the change in kinetic energy associated with the change of spatial distribution of electrons, the antisymmetry of the electronic wave function, and the finite thickness of the surface charge. Without considering the detailed structure of the metal, the authors use a variation method to estimate these corrections. *W. H. Furry (Cambridge, Mass.).*

Caianiello, E. R. On the universal Fermi-type interaction. I, II. *Nuovo Cimento* (9) 8, 534-541, 749-767 (1951).

The first paper examines the question of whether it is possible to ascribe intrinsic parities, as suggested by Yang and Tiomno [*Physical Rev.* (2) 79, 495-498 (1950); these *Rev.* 12, 227], for elementary spin $\frac{1}{2}$ particles for time and space reflection such that all processes not ruled out by charge conservation are forbidden. Using the Wigner-Critchfield interaction it is shown that such an interaction is possible provided one allows two neutrino particle-antiparticle annihilations. Calculations using the interaction are reported. In the second paper a search is made for a universal interaction among all fermions which is invariant under the extended Lorentz group and excludes processes not occurring in nature. Application of certain problematic "simplicity" assumptions yields two mutually exclusive

interactions. The two possible interactions are shown to yield identical answers for free fermions. One of them seems in agreement with all present experimental evidence.

K. M. Case (Ann Arbor, Mich.).

Vrkijan, Vladimir Srečko. *Quelle est la formule de l'analogie du vecteur de Poynting pour le champ des mésons scalaires et pseudoscalaires?* C. R. Acad. Sci. Paris 234, 301-303 (1952).

The known expression of the law of conservation of energy for a field theory derivable from a Lagrangian, enables the author to give the immediate answer to the question posed in the title for the case of neutral scalar, and neutral and charged pseudoscalar meson fields.

A. J. Coleman.

Rideau, Guy. *Sur la formulation des problèmes de diffusion.* C. R. Acad. Sci. Paris 234, 1746-1749 (1952).

Wessel, Walter. *Zur Theorie des Elektrons. II.* Z. Naturforschung 6a, 478-483 (1951).

Previous work of the author [e.g., same Z. 1, 622-636 (1946); these Rev. 9, 128] on the classical kinematics and equations of motion of an electron is developed to lead to the determination of the mass as a function of certain spin invariants. This is done for the case of least spin and depends on the inclusion in the rest-mass of the radiated energy. Spin identities are summarized.

C. Strachan.

Slansky, Serge. *Sur le champ soustractif et le rayon de l'électron.* C. R. Acad. Sci. Paris 234, 602-604 (1952).

Yang, L. M. *A note on the trace of the product of Dirac's matrices.* Philos. Mag. (7) 42, 1333 (1951).

Kohn, Walter. *Variational scattering theory in momentum space. I. Central field problems.* Physical Rev. (2) 84, 495-501 (1951).

Variational principles for scattering problems in a central field are formulated in momentum space and studied in detail. The method allows of generalization to collisions of composite particles, including disintegrations, into more than two end products, which will be discussed in a forthcoming paper.

Author's summary.

Schweber, S. *Perturbation theory and configuration space methods in field theory.* Physical Rev. (2) 84, 1259-1260 (1951).

Michel, Louis. *Théorème sur les invariants formés de quatre fonctions d'onde de Dirac. Application: Section efficace de diffusion nucléon-nucléon.* J. Phys. Radium (8) 12, 793-804 (1951).

The first section is devoted to a review of known results on the tensors which may be formed with two Dirac wave functions. The remainder of the paper is well described in the summary: "It is well-known that with four Dirac wave functions ψ , taken in a definite order, one can form five linearly independent scalars, eight vectors, nine antisymmetric double tensors, eight pseudovectors, and five pseudoscalars. We prove that the permutation of the ψ (in one of these tensors) gives rise to a new tensor of the same type which is a linear combination of the old ones. This theorem proves most useful in effecting calculations. For example, all elements of the S -matrix for a phenomenon involving only four real fermions contain tensors of the types here discussed. Exchange forces arise from permutation of the ψ . As an application the cross-section for nucleon-nucleon

scattering is calculated. The theorem is extended to the thirty-five scalars and thirty-five pseudoscalars formed from six different Dirac wave functions."

A. J. Coleman.

Thirring, W. *Pair creation by mesons.* Proc. Roy. Irish Acad. Sect. A. 54, 205-216 (1951).

The cross-sections for the absorption of a pseudoscalar meson by a nucleon with the emission of a real photon or of a virtual photon which then creates an electron-positron pair are calculated. For low meson velocities the cross-sections are inversely proportional to the velocity. Pair creation is approximately thirty times more probable than photon emission. A very clear exposition of the recent calculational methods is given.

K. M. Case.

Visconti, Antoine. *Sur un modèle classique de particule élémentaire.* C. R. Acad. Sci. Paris 233, 852-854 (1951).

A classical model of an elementary particle is discussed. The meson field surrounding the particle is treated as a compressible fluid.

K. M. Case (Ann Arbor, Mich.).

Battig, A. *Theoretical contributions to the study of the Cherenkov effect.* Univ. Nac. Tucumán. Publ. no. 591, 70 pp. (1951). (Spanish)

Questo lavoro è soprattutto un riassunto degli studi teorici che sono stati compiuti sull'effetto Čerenkov, sia per mezzo dell'elettrodinamica classica, sia per mezzo della teoria quantica del campo. Qualche contributo personale l'autore apporta al problema del passaggio della particella carica da un dielettrico a un altro.

G. Toraldo di Francia

Kwal, Bernard. *Mécanique géométrique non linéaire et la mécanique ondulatoire correspondante.* C. R. Acad. Sci. Paris 234, 508-510 (1952).

A new mechanics of a particle is proposed in which the momentum depends non-linearly on the velocity so that the former is always less than $(mE_0)^{1/2}$ where E_0 is an absolute constant. In the corresponding quantum theory the de Broglie wave length for a particle is less than $h/(mE_0)^{1/2}$ only for unstable states with imaginary energy.

A. J. Coleman (Toronto, Ont.).

van Kampen, N. G. *Contribution to the quantum theory of light scattering.* Danske Vid. Selsk. Mat.-Fys. Medd. 26, no. 15, 77 pp. (1951).

Kramers' [Theorien der Aufbaues der Materie, II, Akademische Verlagsgesellschaft, Leipzig, 1938, pp. 448-460] decomposition of the transverse electromagnetic field into a "proper field" and an "external field" is applied to the scattering and emission of light. The calculations are non-relativistic and restricted to electric dipole radiation. New methods are used to discuss the scattering of incoming light with arbitrary frequency, either in resonance with an absorption line or not. The behavior of Raman scattering inside the line width and the transition to non-resonance are also investigated. Known formulae and their generalizations are obtained.

K. M. Case (Ann Arbor, Mich.).

Thermodynamics, Statistical Mechanics

*Finck, Joseph Louis. *Thermodynamics from a generalized standpoint.* Flatbush Publications, Brooklyn, N. Y., 1951. xi+124 pp. \$4.00.

This book considers situations where the state of a body is specified by three or more independent variables, such

as p , V and a quantity x denoting (say) per cent dissociation. In such situations the concepts of Carnot cycle and entropy become meaningless, and the problem is to develop an adequate thermodynamical theory without these concepts. A complete system is a system completely characterized by sufficiently many independent variables $\alpha_1, \dots, \alpha_n$. Such a system has energy $E(\alpha_1, \dots, \alpha_n)$ and temperature $T(\alpha_1, \dots, \alpha_n)$. Let $A_p = -\partial E / \partial \alpha_p$. Work and heat are distinguished according as $\oint \sum_i A_i d\alpha_i$ is exact or inexact when $k < n$. The Kelvin-Planck and Clausius principles are corollary to the inexactness of this integral. Equilibrium and stability are determined by this integral. Temperature scales for complete systems ($n > 2$) are compared with Kelvin's scale (which assumes $n = 2$). A theory of catalysis is outlined, and low temperature phenomena are discussed. Empirical equations of state, using 3 independent variables, are developed for ammonia, steam, and liquid helium. While some of the reasoning is sketchy, this book has suggestive ideas.

C. C. Torrance (Monterey, Calif.).

Whaples, G. Carathéodory's temperature equations. J. Rational Mech. Anal. 1, 301-307 (1952).

The author shows that Carathéodory's equilibrium equations do not ensure the existence of a numerical temperature, and establishes an adequate and physically usable form for equilibrium assumptions.

C. C. Torrance.

Popoff, Kyrille. Sur la thermodynamique des processus irréversibles. Z. Angew. Math. Physik 3, 42-51 (1952).

The author shows that Onsager's relations remain valid with $L_{ik} = L_{ki}$ when $\Delta S = -\sum g_k x_k - \frac{1}{2} \sum g_{ik} x_i x_k$ and the "forces" $X_k = d^2 x_k / dt^2$.

C. C. Torrance.

Surinov, Yu. A. Solution of a mixed problem on radiative heat transfer for a sphere. Doklady Akad. Nauk SSSR (N.S.) 83, 75-78 (1952). (Russian)

In this paper there is considered the stationary mixed problem for a sphere (filled with a diathermal medium) with a grey surface F which consists of two regions F_1 and F_2 such that $F = F_1 + F_2$. An arbitrary distribution of temperature is given in the region F_1 , the density of the resultant radiation is given on F_2 , and certain optical constants are given for the entire surface F . It is required to determine the fields of the densities of the various types of radiation on the surface as well as in the interior of the sphere. Because of the geometric simplicity of the configuration, the author is able to obtain an exact solution of the problem by elementary methods based on the classification of the types of radiation given in some of his earlier papers [same Doklady 72, 469-472 (1950); these Rev. 12, 467].

H. P. Thielman (Ames, Iowa).

Dutta, M. On a treatment of imperfect gases after Fermi's model. IV. Proc. Nat. Inst. Sci. India 17, 445-466 (1951).

Using methods developed in previous papers [same Proc. 13, 247-252 (1947); 14, 163-168 (1948); 17, 27-37 (1951); these Rev. 10, 275; 13, 196], the author obtains, for various fields of force, equations of state for a mixture of two gases, generalizing results of van der Waals, Lorentz and Dieterici.

C. C. Torrance (Monterey, Calif.).

Johnson, M. H. Diffusion as hydrodynamic motion. Physical Rev. (2) 84, 566-568 (1951).

The author considers a mixture of perfect gases, all having the same local temperature but different densities and pressures; $p_r = n_r kT$, $p = \sum p_r$, $n = \sum n_r$. A phenomenological equation stating the conservation of momentum for each component is postulated:

$$(*) \quad -\frac{\partial p_r}{\partial t} + p_r(F_r - a) + n_r e_r \bar{c}_r \times H = \sum_s \theta_{rs} (\bar{c}_r - \bar{c}_s)$$

(\bar{c}_r is used here for the author's $\langle c_r \rangle_{Av}$, the mean velocity of species r). The right-hand side represents a frictional force between components r and s proportional to their relative velocity. Newton's third law takes the form $\theta_{rs} = \theta_{sr}$. The coefficient θ_{rs} is assumed to be a function of n_r and n_s only. If a single species r is subdivided arbitrarily into components (all at the same temperature T), condition (*) on these subcomponents implies that $\theta_{rs}/n_r n_s$ is a constant for the two species r and s . The formula (*) is shown to be consistent with a certain (primitive) approximation obtained from the Boltzmann equation by Chapman, namely

$$-\frac{\partial p_r}{\partial t} + p_r \left(F_r - \frac{D_r C_0}{Dt} \right) + n_r e_r \bar{c}_r \times H = \sum_s (kT(n_r + n_s)^{-1} n_r n_s / [D_{rs}]_1) (\bar{c}_r - \bar{c}_s).$$

It is incorrectly stated that more subtle diffusion phenomena (e.g. thermal diffusion) cannot be accounted for by a simple modification of (*) keeping the right-hand side a sum of binary terms. The right-hand side will always be a sum of binary terms for a perfect gas if, instead of identifying the acceleration a with $D_c C_0 / Dt$, the acceleration of the mean motion ($\rho C_0 = \sum \rho_r C_r$), it is identified with $D \bar{C}_r / Dt$, the acceleration of the r th component itself. Details can be found in an unpublished thesis by I. Kolodner [New York University, 1950].

H. Grad (New York, N. Y.).

Hirschfelder, Joseph O., Bird, R. Byron, and Spotz, Ellen L. Properties of gases and gas mixtures. Symposium on aerothermodynamics, 30 June 1949. Naval Ordnance Laboratory, White Oak, Md., Rep. NOLR-1134, pp. 1-50 (1950).

The thermodynamic and transport properties of dilute gases and gas mixtures can be determined from the potential energy of interaction of pairs of the various molecular species. Conversely, by assuming a special form for the potential energy function such as the Lennard-Jones potential $E(r) = 4\epsilon[(r_0/r)^{12} - (r_0/r)^6]$ (r is the distance between a pair of molecules and ϵ and r_0 are constants depending on the atomic species) one can derive the values of the viscosity, diffusion constants, equation of state, and other thermodynamic quantities. The various formulae required for calculation of these quantities are reviewed in this report. Finally a large number of tables are presented including: (1) the values of the parameters ϵ and r_0 of the above formula for a long series of gases; (2) certain functions from which one can determine second virial coefficients and other thermodynamic quantities; (3) viscosity coefficients of mixtures; (4) coefficients of self diffusion and thermal diffusion.

E. W. Montroll (College Park, Md.).

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